Geometric Langlands & Hitchin system

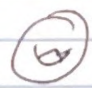
$/\mathbb{C}$ Global geom. Langlands

X smooth projective curve (connected)

$=$ compact Riemann surface

$/\mathbb{R}$ sphere with $\underbrace{g}_{\text{genus}}$ handles

$g=0$  $\mathbb{P}^1 = \mathbb{C}\cup\infty$

$g=1$  $\mathbb{C}/\Lambda$

$\{y^2+x^3+ax+b\}\cup\infty \subset \mathbb{P}^2$
elliptic curve

$g=2$  hyperelliptic

vector bundle of rank r $\quad E \xrightarrow{\pi} X \ni x$

$\pi^{-1}(x) \approx \mathbb{C}^r$

$E \times_X E \xrightarrow{+} E$ add in the fiber

$\forall x \in X \quad x \in U \subset X$

$\pi^{-1}(U) \approx \mathbb{C}^r \times U$ with respect to add/scalar mult. on fibers

$E \xrightarrow{\pi} X$ $\quad E(\mathcal{U}) = \{s : \mathcal{U} \to E,\ \pi \circ s = id_{\mathcal{U}}\}$

locally free $\mathcal{O}_X$-modules of finite rank

$r=1$ line bundle $L \to X$

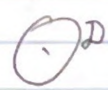
$H^0(X, \mathcal{O}_X) = \{X \to \mathbb{C}\} = \mathbb{C}$

$H^0(X, L)$ can be large

e.g. $\Omega_X = T^*X \quad r=1$

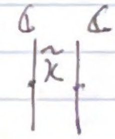
$\dim H^0(X, \Omega_X) = g$

$x \in X \quad n \in \mathbb{Z}$



$\mathbb{C} \times (X - x)$

glue



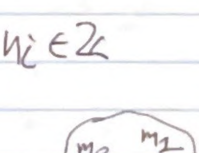
$\mathbb{C} \times D$

$\mathcal{O}_X(nx) \quad H^0(\mathcal{U}, \mathcal{O}_X(nx)) = \{s : \mathcal{U} - x \to \mathbb{C}$, have pole order at most n at $x\}$

Thm. $L \xrightarrow{\mathbb{C}} X \quad \exists\, x_i \in X \quad n_i \in \mathbb{Z}$

$L \cong \bigotimes_{i=1}^{n} \mathcal{O}_X(n_i x_i)$

may not unique



but $\deg L = \sum n_i \in \mathbb{Z}$ depends on L only

$g=0 \quad \infty \in \mathbb{P}^1 \quad \forall L \quad L \simeq \mathcal{O}_{\mathbb{P}^1}(n\cdot\infty)$
$n = \deg L$

$X = \mathcal{E} \quad g=1$

Fix $e \in \mathcal{E}$ $\quad \forall L \xrightarrow{e} \mathcal{E} \quad \exists x \in \mathcal{E}$

$$L \simeq \mathcal{O}_{\mathcal{E}}(x) \otimes \mathcal{O}_{\mathcal{E}}(ne)$$
$$n = \deg L - 1$$

$\mathrm{Pic}\, X$ = moduli space of line bundles

$\Vert$ $\quad$ pts $\Leftrightarrow$ line bundles

$\coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d(X)$

non-canonically $\mathrm{Pic}^d(X) \simeq \mathrm{Pic}^{d'}(X)$

$$\mathrm{Jac}(X) := \mathrm{Pic}^0(X)$$

e.g. $\mathrm{Pic}^1(\mathcal{E}) = \mathcal{E}$ canonically iso-
$L \leftrightarrow x$

$\mathrm{Pic}^d(X) \underset{\mathbb{R}}{\simeq} (S^1)^{2g}$ $\quad$ abelian variety
$\Vert$
proj. group $\otimes$

$$\mathrm{Pic}^d(X) \not\simeq X$$

$$\mathrm{Pic}(\mathrm{Pic}^d(X)) \simeq \mathrm{Pic}^d(X)$$
$\uparrow$
parametrices trivial line bundles on itself

Hitchin system

V, $\varphi : V \to V$ $\quad V \simeq \mathbb{C}^r$

$\det(\varphi - \lambda \, \mathrm{Id}) = 0$

eigenvalues $\lambda_1, \cdots, \lambda_r$

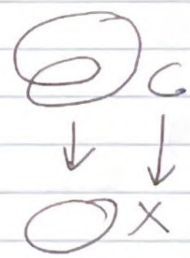
eigenspaces $l_i \subset V$

$\varphi(z) : V \to V \quad z \in X$

$$\begin{array}{c} \{\det(\varphi(z) - \lambda \mathrm{Id}) = 0\} \\ \downarrow r \to 1 \\ X \end{array}$$

r to 1 cover

Spectral cover



v.b.

$E \to X \quad \varphi : E \to E \otimes \Omega_X$ Higgs field

(E, φ) Higgs bundle if φ

$\operatorname{tr}\varphi$ - trivialize E locally $E \simeq \mathbb{C}^r \times U$

$$\begin{array}{ll} \operatorname{tr}\varphi & \\ \| & \\ (\varphi_{ij}) & \varphi_{ij} \in \Omega_X(U) \\ \| & \quad \| \\ \sum \varphi_{ii} & \in H^0(U, \Omega_X) \end{array}$$

Thm. $\operatorname{tr}\varphi$ does not depend on the trivilization.

Pf. change the trivilization $(\varphi_{ij}) \Rightarrow A(z)(\varphi_{ij})A(z)^{-1}$

does not change the trace $\boxtimes$

$\varphi: E \to E \otimes \Omega_X \qquad tr\varphi \in H^0(X, \Omega_X)$

$tr\varphi^2 \in H^0(X, \Omega_X \otimes \Omega_X)$

....

$tr\varphi^r \in H^0(X, \Omega_X^{\otimes r})$

$Bun_r(X)$ = moduli space of rank r v.b. on X

$Bun_1(X) = Pic(X)$

Artin stack

$$\begin{tikzcd} Higgs_r(X) = \{(\mathcal{E},\varphi) : \varphi: E \to E\otimes\Omega_X\} \arrow[d] \arrow[r, phantom, "\cong"] & T^*Bun_r(X) \arrow[dl] \\ Bun_r(X) & \end{tikzcd}$$

completely integrable system

$$Higgs_r(X) \xrightarrow{H} \bigoplus_{i=1}^{r} H^0(X, \Omega_X^{\otimes i}) = Hitch_r(X) \ni b$$

vector space

$$(\mathcal{E},\varphi) \mapsto (tr\varphi, tr\varphi^2, \cdots)$$

parametrize the covers

Hitchin ~~B~~ fibrations

$$(\mathcal{E},\varphi) \mapsto \{\det(\lambda Id - \varphi) = 0\} = C \xrightarrow{r:1} X$$

$H^{-1}(b) \cong Pic(C_b) \leftarrow$ fiber is an abelian variety

$\exists$ bad fibers

$\mathrm{Higgs}_r(X) = (\mathcal{E}, \varphi)$

$$\downarrow$$

$\ell \in \mathrm{Pic}(C_b)$

$L_\ell \in \mathrm{Pic}(C_b)$

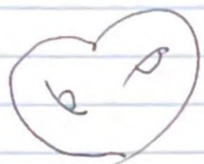
point of $\mathrm{Higgs}_r(X)$ gives a (generalized) line bundle on the same Hitchin fiber

$\circlearrowleft D^b(\mathrm{Higgs}_r(X))$

Langlands for Hitchin systems

X smooth proj. $/\mathbb{C}$

$g(X) > 2$ (can do $X = \mathbb{P}^1$)



$$\mathrm{Bun}_{r,d}(X) = \{ E \xrightarrow[\mathbb{C}^r]{} X \,.\, \deg \Lambda^r E = d \}$$

$\uparrow$ stack $\qquad$ pts = iso-classes of v.b.

$$\dim \mathrm{Bun}(X) = r^2(g-1)$$

$$\mathrm{Higgs}_{r,d}(X, \ell) = \{ (E, \varphi) : E \to E \otimes \ell \}$$
(usually $\ell = \Omega_X$)

$$\ell = \mathcal{O}_X(x_1 + x_2 + \cdots + x_k) \qquad \varphi : E \to E(x_1 + \cdots + x_k)$$

$$\mathrm{Higgs}_{r,d}(X) = T^* \mathrm{Bun}_{r,d}(X)$$

E v.b. $\qquad T_E \mathrm{Bun}_{r,d}(X) = H^1(X, \mathrm{End}(E))$

$$T^*_E \mathrm{Bun}_{r,d}(X) = H^0(X, \mathrm{End}(E) \otimes \Omega_X) = \{ \varphi : E \to E \otimes \Omega_X \}$$

$$\begin{array}{ccc}
\mathrm{Higgs}_{r,d}(X) & & (E, \varphi) \\
\downarrow \text{Integrable System} & & \downarrow \\
\bigoplus_{i=1}^{r} H^0(X, \Omega_X^{\otimes i}) & & \{ \det(\lambda \mathrm{Id} - \varphi) = 0 \} \\
\wr\!\wr & & \\
\mathrm{Spec}\, H^0(\mathrm{Higgs}, \mathcal{O}_{\mathrm{Higgs}}) & &
\end{array}$$

Algebraic completely integrable system

(Y, ω) - symplectic variety $\dim_{\mathbb{C}} Y = 2d$

$$\begin{array}{l} Y \\ \pi \big\downarrow \leftarrow \text{Lagrangian fibration} \\ b \in V \text{ (vector space)} \\ h \big\downarrow \quad \omega|_{\pi^{-1}(b)} = 0 \\ \mathbb{C} \end{array}$$

$\omega \in H^0(Y, \Lambda^2 \Omega_Y)$

$\dim V = d$

Assume that π is proper (projective)

Smooth fibers will be abelian varieties

$dh(\xi) = \omega(\xi_h, \xi)$ determines $! \, \xi_h \leftarrow$ tangent to the fibers

$[\xi_h, \xi_{h'}] = 0 \Leftrightarrow \xi_h(h') = 0$

$\{\det(\lambda Id - \varphi) = 0\} = C_{(E, \varphi)}$
$\cap$
$T^* X$

$$\begin{array}{c} \mathrm{Higgs}_{r,d}(X) \\ \big\downarrow H \\ b \in \bigoplus_{i=1}^{r} H^0(X, \Omega_X^{\otimes i}) \end{array}$$

$H(\mathcal{E}, \varphi_{\mathcal{E}}) = b$
$C_{\mathcal{E}, \varphi} = C_b$

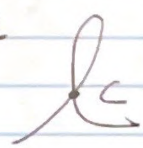
$H^1(b) = \{ \ell \to C_b, \deg \ell = d^1 \} \simeq Jac(C_b)$

provided C_b is smooth

$Jac(Jac(C_b)) \simeq Jac \, C_b$

If C_b not smooth

$$\mathrm{Jac}(C_b) \text{ not compact}$$
$$\cap$$
$$\overline{\mathrm{Jac}}(C_b) = \left\{ \begin{array}{l} \text{sheaves on } C_b \text{ of generic rank } 1 \\ \text{no torsion} \end{array} \right\}$$

Example 

$$\overline{\mathrm{Pic}}^{-1}(C) = \{\mathcal{O}_X(-x),\ x \in C\}$$
$$\cong$$
$$\overline{\mathrm{Jac}}(C) \simeq C$$
$$\cup$$
$$y^2 = x^2(x+1) \qquad \mathrm{Jac}(C) \simeq C - p \simeq \mathbb{C}^\times$$

$$\mathrm{Jac}(\mathrm{Jac}(C)) \simeq \mathrm{Jac}(C)$$

For any Hitchin spectral curve, expect

$$\overline{\mathrm{Jac}}(\overline{\mathrm{Jac}}(C)) \simeq \overline{\mathrm{Jac}}(C)$$

- If C is integral, known Arinkin (2010)
- If C is reduced (e.g. ∞), ?
- If C is non-reduced (e.g. $x^3=0$), ??

$$\begin{array}{ccc} \mathrm{Higgs}(X) & & \{(\mathcal{E},\phi) : \phi^r = 0\} \text{ global nilpotent cone} \\ \downarrow & & \downarrow \\ b \in \bigoplus H^0(X, \Omega_X^{\otimes i}) & & 0 \end{array}$$

$$\det(\phi - \lambda \mathrm{Id}) = 0 \Leftrightarrow \phi^r = 0$$

$\circlearrowleft D^b(H^1(\mathcal{C}))$ from $\overline{\mathrm{Jac}}(C) \simeq C$

$\cancel{X \otimes} E \overset{\nabla}{\rightarrow} Y$

$E \xrightarrow{\nabla} E \otimes \Omega_X$ (morphism of sheaves)

$\nabla(s_1+s_2) = \nabla(s_1) + \nabla(s_2)$

$\nabla(fs) = f\nabla s + \cancel{s \otimes df}\ \varepsilon\, s \otimes df \quad \varepsilon \in \mathbb{C}$

$\begin{cases} \varepsilon = 1 & \text{connection} \\ \varepsilon = 0 & \text{Higgs field} \end{cases}$

[$\varepsilon \neq 0$ $\quad \frac{1}{\varepsilon}\nabla$ is a connection on E]

$\widetilde{\mathrm{Conn}}(X)$ – moduli space of ε-connections for $\forall \varepsilon$

$\downarrow$

$\varepsilon \in \mathbb{C}$

0-fiber $\mathrm{Higgs}(X)$

$\varepsilon \neq 0$ $\quad$ ε-fiber $\approx \mathrm{Conn}(X)$

$\mathrm{Conn}(Y) = \{E \xrightarrow{\nabla} E \otimes \Omega_Y\}$ – stack

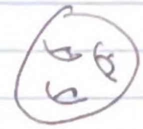
$\cap$ $\qquad$ category

$\mathcal{D}_Y$-modules $\quad$ E can be a quasi-coh sheaf

$Y = \mathbb{C}$ $\qquad \mathbb{C}[\delta] \qquad \nabla\delta = \delta^2 \otimes dx$

$\downarrow \qquad x\delta = 0$

$0 \qquad \partial_z \delta = \delta^2$

$(E, \nabla) \in \mathrm{Conn}(X)$ $\qquad X =$ 

$\rightsquigarrow$

$A_{E,\nabla} \in \mathcal{D}_{\mathrm{Bun}(X)}\text{-module}$ $\qquad \varepsilon \neq 0$

automorphic sheaf

$(E, \varphi) \in \mathrm{Higgs}(X)$ $\qquad \varepsilon = 0$

$\rightsquigarrow$

$A_{E,\varphi} \in \mathrm{Coh}(T^*\mathrm{Bun})$

$(F, \nabla) \in \mathcal{D}_Y\text{-mod}$ $\qquad (\nabla^2 = 0)$ flat condition

$\nabla : F \to F \otimes \Omega_Y$ $\qquad$ for curve X is trivial

ξ vector fields on Y

$s \in F \qquad \xi s \in F$

$\varepsilon[\xi_1, \xi_2]s = \xi_1 \xi_2 s - \xi_2 \xi_1 s$

$\varepsilon = 0 \qquad 0 = \xi_1 \xi_2 s - \xi_2 \xi_1 s$

$\Rightarrow F \in \mathrm{Coh}(T^*Y)$

$\mathcal{O}(T^*Y) = \mathrm{Sym}^{\cdot}\, TY$

$X \qquad \mathrm{Higgs}_\gamma(X) \rightsquigarrow \mathrm{QCoh}(\mathrm{Higgs}_\gamma(X)) = \mathrm{QCoh}(T^*\mathrm{Bun}_\gamma(X))$

$\mathrm{Conn}_\gamma(X) \rightsquigarrow \mathcal{D}\text{-mod}(\mathrm{Bun}_\gamma(X)) \leftarrow$ Hecke functors

$(E, \nabla) \Rightarrow$ Hecke eigensheaf

$\mathrm{Conn}^\varepsilon_\gamma(X) \rightsquigarrow \mathcal{D}\text{-mod}^\varepsilon(\mathrm{Bun}_\gamma(X))$

$$\mathcal{D}\text{-mod}^\varepsilon(Y) = \begin{cases} \mathcal{D}\text{-mod}(Y), & \varepsilon = 1 \\ \mathrm{QCoh}(T^*Y), & \varepsilon = 0 \end{cases}$$

$V \simeq \mathbb{C}^r = \mathrm{Spec}\,\mathbb{C}[x_1, \cdots, x_r]$

$\mathrm{QCoh}(V) \approx \mathbb{C}[x_1, \cdots, x_r]$-module

$\{ L - \mathbb{C}$-v.s. with commuting action of x_i-s$\}$

$V^* \otimes L \to L$

$V^* \otimes V^* \otimes L \rightrightarrows L$

$E \to Y \qquad E$ - total space, sheaf

$$\mathrm{QCoh}(E) = \left\{ \begin{array}{l} F \in \mathrm{QCoh}(Y) \\ E^* \otimes F \to F \\ s_1(s_2 f) = s_2(s_1 f) \end{array} \right\}$$

$T^*Y \to Y$

$TY \otimes F \to F \qquad (\xi_1\xi_2 - \xi_2\xi_1)f = \varepsilon[\xi_1, \xi_2]f$

$\xi_1(\xi_2 f) = \xi_2(\xi_1 f) \qquad \mathcal{D}^\varepsilon_{\mathrm{mod}}(Y)$

$(\xi_1\xi_2 - \xi_2\xi_1) f = 0$

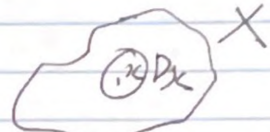
$\mathrm{Conn}_r(X)$ — $\mathrm{Rep}(\pi_1(X), GL_r(\mathbb{C}))$ Riemann-Hilbert correspondence

$\mathrm{Rep}(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), GL_r(\mathbb{C}))$

$\mathrm{Bun}'_r(X) = \{E \to X, \Lambda^r E \underset{\gamma}{\xrightarrow{\sim}} \mathcal{O}_X\}$

Fix $x \in X$ $\quad E|_{X-x} \approx \mathbb{C}^r \times (X - x)$

$X = (X-x) \sqcup D_x$



$D_x \approx \mathrm{Spec}\, \overset{\mathcal{O}}{\mathbb{C}[[t]]}$

$D_x - x = \mathrm{Spec}\, \mathbb{C}((t))\ K$

$g \in SGL_r(\mathbb{C}((t))) = SGL_r(K)$

$\mathrm{Bun}'_r(X) = SGL_r(X-x) \backslash SGL_r(K) / SGL_r(\mathcal{O})$

$SL_r(\mathbb{Q}_p) / SL_r(\mathbb{Z}_p)$

"automorphic functions"

$\mathrm{Mod}^r_{X,x} = \{(E, E', s) \mid E|_{X-x} \underset{s}{\cong} E'|_{X-x}\}$

$\cancel{\mathcal{D}\text{-mod}^r_{X,x} \leftarrow \mathcal{D}\text{-mod}(\mathrm{Bun}}$

$\mathcal{D}\text{-mod}(\mathrm{Mod}^r_{X,x}) \leftarrow \mathcal{D}\text{-mod}(\mathrm{Bun}_X)$

$$\begin{tikzcd} & (E, E', s) \in \mathrm{Mod}^r_{X,x} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ E \in \mathrm{Bun}_X & & E' \in \mathrm{Bun}_X \end{tikzcd}$$

$$F \in \mathcal{D}\text{-mod}(\mathrm{Mod}^{\vee}_{X,x}) \quad A \in \mathcal{D}\text{-mod}(\mathrm{Bun}_X)$$

$$p_{1,*}(F \otimes p_2^* A) = V \otimes F$$

$$\{E \hookrightarrow E' : E'/E \simeq \mathcal{O}_x^{\oplus i}\} = \mathcal{M}_i \overset{\iota}{\subset} \mathrm{Mod}^{\vee}_{X,x}$$

$$F_i := \iota_* \mathcal{M}_i \qquad D = \mathrm{Spec}\, \mathbb{C}[[t]]$$

$$\mathrm{Mod}^{\vee}_D = \{(E, E', s) ; E, E' \underset{\vee}{\to} D, s : E|_{D-x} \simeq E'|_{D-x}\}$$

$$\| \qquad G = GL_r \qquad D - x = \mathring{D} = \mathrm{Spec}\, K$$

$$G(\mathcal{O}) \backslash G(K) / G(\mathcal{O})$$

$G(K)/G(\mathcal{O})$ affine Grassmannian

$$\mathcal{D}\text{-mod}(\mathrm{Mod}^{\vee}_D) = \mathcal{D}\text{-mod}(G_{\mathcal{O}} \backslash \mathrm{Gr}_G)$$

$$= \mathcal{D}\text{-mod}^{G_{\mathcal{O}}}(\mathrm{Gr}_G)$$

$$G \curvearrowright Y$$

$$\mathcal{D}\text{-mod}(G \backslash Y) = \mathcal{D}\text{-mod}^G(Y)$$

$\|$ geometric Satake

$$\mathrm{Rep}(\check{G})$$

$$G = GL_r \quad \check{G} = GL_r \qquad \|$$

$\mathrm{QCoh}(\{(E, \nabla), (E', \nabla')$: bundles with connections on D
$s : (E, \nabla)|_{\mathring{D}} \simeq (E', \nabla')|_{\mathring{D}}\})$

$$D^b(\mathrm{Higgs}_G(X)) \simeq D^b(\mathrm{Higgs}_{\check{G}}(D_x))$$

$$\{(\mathcal{E}_i, \varphi_i) \to D,\ i=1,2$$

$$(\mathcal{E}_1,\varphi_1)|_{D-x} \simeq (\mathcal{E}_2,\varphi_2)|_{D-x}\}$$

$$\| $$

$$D^b(T^*(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}))) \simeq D^b(T^*(\check{G}(\mathcal{O})\backslash \check{G}(K)/\check{G}(\mathcal{O})))$$

local Hitchin-Langland's correspondence

Torsors = Principal bundles

$$\begin{array}{c} E \\ G \circlearrowright \downarrow \pi \\ X \end{array}$$

$\pi(ge) = \pi(e)$

$\forall e_1, e_2 \quad \pi(e_1) = \pi(e_2) \Rightarrow \exists!\, g \in G$
$e_2 = g e_1$

Ex.

$$\begin{array}{cc} z & \mathbb{C}^* \\ \downarrow & \downarrow \mu_2 \\ z^2 & \mathbb{C}^* \end{array}$$

$G = \mathbb{Z}/2\mathbb{Z} = \mu_2 = \{\pm 1\} \subset \mathbb{C}^*$

affine group scheme/$\mathbb{C}$ of finite type

Def. /$\mathbb{C}$ linear algebraic group is a Zariski closed subgroup in $GL_n(\mathbb{C}) = \{A \in Mat_{n\times n}(\mathbb{C}) \mid \det A \neq 0\}$
(also a subvariety)

Ex. $GL_n(\mathbb{C})$, $SL_n(\mathbb{C}) := \{A : \det A = 1\}$

$O_n := \{A : A^t A = Id\}$

$SO_n := O_n \cap SL_n$

$G_a = (\mathbb{C}, +) \qquad PGL_n := GL_n/\text{center}$

$GL_1 = G_m = \mathbb{C}^*$

Thm. Every linear algebraic group is smooth.

Pf. (i) $\forall x, y \in G \Rightarrow \exists\, G \xrightarrow[a]{\sim} G \quad a(x) = y$

(ii) A homogenous complex variety is smooth.

(i) $a: z \mapsto z \cdot (x^{-1}y)$ $\qquad a^{-1} = \cdots$

(ii) $\exists\ U \underset{\text{open}}{\subset} G$ s.t. U is smooth, $y \in U$, $x \in G$

$a: y \mapsto x$ $\qquad a(U) \underset{\text{open}}{\subset} G$, $y \in a(U)$ smooth

$\blacksquare$

Remark ① G is a Zariski closed subset.
(every group $/\mathbb{C}$ is reduced)

② S – a scheme

Group scheme $\quad G \times_S G \xrightarrow{m} G$

Def. Let G be as above, X a complex variety.
Then a G-torsor over X is the following data

(i) $E \xrightarrow{p} X$, where p is smooth

(ii) action $\quad G \times E \xrightarrow{a} E$

$$\begin{tikzcd}
G\times G\times E \arrow[r, "(m{,}id)"] \arrow[d, "(id{,}a)"'] & G\times E \arrow[d, "a"] \\
G\times E \arrow[r, "a"'] & E
\end{tikzcd}$$

such that (a)
$$\begin{tikzcd}
G\times E \arrow[r, "a"] \arrow[dr, "pr_2"'] & E \arrow[d, "p"] \\
& X
\end{tikzcd}$$

(b) $G\times E \underset{\approx}{\xrightarrow{(pr_2, a)}} E \times_X E$ Iso-

$(g, e) \mapsto (e, ge)$

$\boxed{G = Gl_r}$

$E \xrightarrow[r]{} X$ vector bundle

$F := \{x \in X, e_1, \cdots, e_r \in E_x, \{e_1, \cdots, e_r\} \text{ is a basis}\}$

$$\cap$$

$$\underbrace{E \times_X E \times_X E \times_X \cdots \times_X E}_{r}$$

$Gl_r \curvearrowright F$ matrix $\curvearrowright$ basis $\rightsquigarrow$ new basis

$\boxed{O_r}$ Quadratic bundle $E \xrightarrow[r]{} X$ $E \times_X E \xrightarrow[Q]{} \mathbb{C}$

$Q|_{E_x \times E_x}$ is a non-degenerate quadratic form

similar F as above, while $\{e_1, \cdots, e_r\}$ as an orthogonal basis

R-ring $O_{n,R} \subset Gl_{n,R}$

$Q : R^n \times R^n \to R$

$(R^n, Q) \rightsquigarrow O_{n,R}$ - torsor

underlying vector bundle is trivial

$H \hookrightarrow X \leftarrow$ variety $\qquad X \xrightarrow[H]{} X/H$

group free

principal H-bundles ($=H$-torsor)

$H \subset G \qquad G \xrightarrow[H]{} G/H$

linear algebraic groups

G - reductive $\rightsquigarrow \check{G}$ - Langlands dual

$SL_n \qquad PSL_n$

$GL_n \qquad GL_n$

$D^b(T^*\mathrm{Bun}_r(X))$ 2

$\parallel$

$D^b(T^*\mathrm{Bun}_{GL_n}(X)) \xrightarrow{\sim} D^b(T^*\mathrm{Bun}_{GL_r}(X))$

$D^b(T^*\underbrace{\mathrm{Bun}_G(X)}_{\text{moduli of } G\text{-bundles on } X}) \xrightarrow{\sim} D^b(T^*\mathrm{Bun}_{\check{G}}(X))$

Generalities ① $f^*E \to E \qquad E \times_X Y = \{(e,y) : p(e)=f(y)\}$

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "G"'] & E \arrow[d, "G"] \\ Y \arrow[r] & X \end{tikzcd}$$

② iso:

$$\begin{tikzcd} E_1 \arrow[rr, "\iota"] \arrow[dr] & & E_2 \arrow[dl] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} G \times E_1 \arrow[r, "(id,\iota)"] \arrow[d] & G \times E_2 \arrow[d] \\ E_1 \arrow[r, "\iota"] & E_2 \end{tikzcd}$$

③ $G \times X \xrightarrow[p_2]{} X$ is a G-torsor (isomophic to this called trivial)

④ $E \xrightarrow{G} X$ is trivial $\Leftrightarrow$ $E \to X$ has a section

Pf. "$\Rightarrow$" $\quad x \mapsto (e, x)$ is the section

"$\Leftarrow$" $\quad E \underset{G}{\overset{s}{\rightleftarrows}} X$

$G \times X \to E$ isomorphism

$(g, x) \mapsto g s(x)$

Naively $\quad e \in E \quad x = p(e) \quad p(e) = p(s(x))$

$\exists!\, e = g s(x)$

$\blacksquare$

Local triviality

$$\begin{tikzcd} E|_U \arrow[r] \arrow[d] & E \arrow[d, "G"] \\ x \in U \arrow[r, hook, "j"] & X \end{tikzcd}$$

Zariski

$E|_U = j^* E$

Def. E is Zariski locally trivial if $\forall x \in X$

$\exists$ a Zariski open neighborhood U s.t. $E|_U$ is trivial.

$$\begin{array}{cc} \mathbb{C}^* & z \\ \downarrow & \downarrow \\ \mathbb{C}^* & z^2 \end{array}$$

not locally trivial

Every GL_r-torsor is trivial

$\updownarrow$

vector bundle

$\updownarrow$

locally free sheaf

$\updownarrow$

Zariski locally trivial

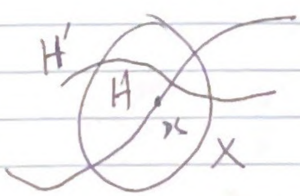
Serre 1958 (Fibrés) $\quad G \to G/H$

Thm (Colliot-Thélène) [Grothendieck-Serre Conj.]

$E \xrightarrow{G} X$, X is smooth. Assume rationally trivial

$\exists\, U \subset X$, $U \neq \phi$ $\quad E|_U$ is trivial

$\Rightarrow$ Then Zariski locally trivial.



Thm. (F-Panin, 2012) $\quad E_1 \xrightarrow{G} X \xleftarrow{G} E_2$ rationally iso-

$\Rightarrow$ Zar- locally trivial.

$G \subset GL_{r,\mathbb{C}}$

$E \xrightarrow[G]{} X$

$$\begin{tikzcd}
G \times E \arrow[r, "act"] \arrow[dr] & E \arrow[d] \\
& X
\end{tikzcd}$$

$$G \times E \xrightarrow{\sim} E \times_X E$$

Prop. let $E = G \times X$ be trivial, then $\mathrm{Aut}(E) = \mathrm{Mor}_{\mathbb{C}}(X, G)$

Pf. $t : X \to G$

$$(g, x) \overset{A_t}{\longmapsto} (gt, x)$$

$$\begin{tikzcd}
g'(g,x) = (g'g, x) \arrow[r, "A_t"] \arrow[d, "A_t"] & (g'gt, x) \\
g'(gt, x) \arrow[ur, equal] &
\end{tikzcd}$$

$$\begin{tikzcd}
(e,x) & G \times X \arrow[r, "a"] & G \times X \arrow[r, "p"] & G \\
x \arrow[u, mapsto] & X \arrow[u] \arrow[urr, "M_a"'] & &
\end{tikzcd}$$

$$M_a(x) = p(a(e,x))$$

$E \xrightarrow{G} X$

Assume Zariski locally trivial

$X = \bigcup_\alpha U_\alpha$

$$\forall \alpha \qquad \begin{tikzcd} G \times U_\alpha \arrow[rr, "\sim", "S_\alpha"'] \arrow[dr] & & E \times_X U_\alpha \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

On $U_\alpha \cap U_\beta =: U_{\alpha\beta}$ $\qquad \underline{U} = \{U_\alpha\}$

$$\begin{tikzcd} G \times U_{\alpha\beta} \arrow[r, "S_\beta|U_{\alpha\beta}"] & E \times_{U_{\alpha\beta}} E \arrow[l, bend left, "S_\alpha^{-1}|U_{\alpha\beta}"] \end{tikzcd}$$

$U_\alpha \cap U_\beta \cap U_\beta =: U_{\alpha\beta\gamma}$

Now $S_\alpha^{-1}|_{U_{\alpha\beta}} \circ S_\beta|_{U_{\alpha\beta}} = g_{\alpha\beta} \in Mor_{\mathbb{C}}(U_{\alpha\beta}, G)$

Lemma. $\{g_{\alpha\beta} : U_{\alpha\beta} \to G\} \in Z^1(\underline{U}, G)$

Pf. $\begin{cases} \forall \alpha : & g_{\alpha\alpha} = 1 \\ & \Uparrow \\ \forall \alpha, \beta, \gamma & g_{\alpha\beta}|_{U_{\alpha\beta\gamma}} \circ g_{\beta\gamma}|_{U_{\alpha\beta\gamma}} = g_{\alpha\gamma}|_{U_{\alpha\beta\gamma}} \\ & \Downarrow \\ \forall \alpha, \beta & g_{\alpha\beta}^{-1} = g_{\beta\alpha} \end{cases}$

∎

$Z^1(\underline{U}, G) \rightsquigarrow G$ - torsor over X

$Y_\alpha := G \times U_\alpha \to U_\alpha$

iso $\underbrace{Y_\alpha \times_{U_{\alpha\beta}} U_\alpha}_{= Y_{\alpha\beta}} \simeq \underbrace{Y_\beta \times_{U_{\alpha\beta}} U_\beta}_{= Y_{\beta\alpha}}$

$Y_{\alpha\beta} \subset Y_\alpha$

$\wr\!\text{S}$

$Y_{\beta\alpha} \subset Y_\beta$

Obtain E by gluing Y_α's over $Y_{\alpha\beta}$

$$E \to X$$

$$G \times E \to X$$

Claim. $E \to X$ is a G-torsor

$$\begin{tikzcd} G \times E \arrow[r, "\sim"] \arrow[dr] & E \times_X E \arrow[d] \\ & X = \bigcup_\alpha U_\alpha \end{tikzcd}$$

After pullback to U_α

$$G \times G \times U_\alpha \to (G \times U_\alpha) \times_{U_\alpha} (G \times U_\alpha)$$

$$\begin{tikzcd} g_{\alpha\beta} \rightsquigarrow & E \arrow[d] \arrow[r] & X \\ g'_{\alpha\beta} \rightsquigarrow & E' \arrow[ur] & \end{tikzcd}$$

Restrict

$$\begin{tikzcd} G \times U_\alpha \simeq E|U_\alpha \arrow[d, "t_\alpha"] \arrow[d, shift left=8ex, "\sim"] \arrow[dr] & \\ G \times U_\alpha \simeq E'|U_\alpha \arrow[r] & U_\alpha \end{tikzcd}$$

$$t_\alpha \in Mor_k(U_\alpha, G)$$

$$g'_{\alpha\beta} = t_\alpha g_{\alpha\beta} t_\beta^{-1} \quad (*)$$

Def. $g_{\alpha\beta} \sim g'_{\alpha\beta}$ if $\exists\, t_\alpha : U_\alpha \to G$ s.t. (☆)

$$H^1_{zar}(\underline{U}, G) = Z^1(\underline{U}, G)/\sim$$

Thm. $H^1_{zar}(\underline{U}, G)$ is in a natural bijection with

pointed set

the set of isoclasses of G-torsor/X

trivial on U_α

can't define H^2_{zar}?

Prop. Every torsor is locally trivial in smooth topology.

$\forall\, E \xrightarrow[p]{G} X \Rightarrow \exists$ a smooth surjective $Y \twoheadrightarrow X$

$$\begin{array}{c} E \times_X Y \\ \downarrow G \\ Y \end{array}$$ is trivial as a G-torsor

$Y_\alpha \xrightarrow[\text{smooth}]{} X$

$Y = \coprod_\alpha Y_\alpha \underset{\text{surj.}}{\twoheadrightarrow} X$

$Y := E \xrightarrow{p} X$

$$\begin{tikzcd} E \times_X E \arrow[r] \arrow[d, equal] & E \arrow[d, equal] \\ E \times_X Y \arrow[r] & Y \end{tikzcd}$$

diagonal section $e \mapsto (e, e)$

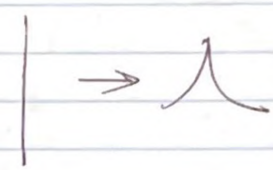
$G\times E \xrightarrow{\sim} E\times_X E$ (def) ∎

Cor. Every torsor is étale locally trivial

Prop. A torsor is trivial iff it has a section.

Pf. "⇒" trivial

"⇐"
$$\begin{tikzcd} E \arrow[d, shift right, "p"'] & G\times X \arrow[l] \arrow[dl] \\ X \arrow[u, shift right, "s"'] & \end{tikzcd}$$
$(g,x) \mapsto g\,s(x)$



$$\begin{tikzcd} Y\times_{X_2} X_1 \arrow[r, "\tilde f"] \arrow[d] & Y \arrow[d, "\text{smooth}"] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd}$$

f iso $\Leftrightarrow$ $\tilde f$ iso

Prop. $\begin{tikzcd} y\in Y \arrow[d] \\ x=f(y)\in X \end{tikzcd}$ smooth morphism of $\mathbb{C}$-schemes

$$\begin{tikzcd} \exists X' \arrow[r] \arrow[dr, "\text{étale}"'] & Y \arrow[d, "f"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} x' \arrow[r, mapsto] \arrow[dr, mapsto] & y \arrow[d, mapsto] \\ & x \end{tikzcd}$$

$$\begin{tikzcd} E\times_X X' \arrow[r] \arrow[d] & E\times_X Y \arrow[r] \arrow[d] & E \arrow[d, "G"] \\ X' \arrow[r] \arrow[rr, bend right, "\text{étale}"'] & Y \arrow[r, two heads] & X \ni x \end{tikzcd}$$

$$\begin{tikzcd} x' \arrow[r, mapsto] \arrow[rr, bend right, leftrightarrow, "ex"'] & y \arrow[r, mapsto] & x \end{tikzcd}$$

Do it for all $x \in X$

$$\coprod X'_x \overset{\text{étale}}{\twoheadrightarrow} X \qquad U_x := \varphi_x(X'_x) \underset{\text{open}}{\subset} X$$

$$\exists\, x_1, \dots, x_n$$

$$\bigcup U_{x_i} = X$$

$$X' = \coprod_i X'_{x_i} \twoheadrightarrow X$$

$\blacksquare$

$$H^1_{zar}(X, G) = \varinjlim H^1_{zar}(\underline{U}, G)$$

all Zariski locally trivial

$$H^1_{ét}(X, G) = \varinjlim H^1_{étale}(\underline{U}, G)$$

① Non-abelian cohomology

② $GL(r)$ - torsors

③ torsors on $\mathbb{P}^2, \mathbb{A}^2$

④ Grothendieck-Serre conjecture

$$H^1_{Zar}(X, G) = \varinjlim H^1_{\check{C}ech\,Zar}(X, G)$$

$$G \subset GL(n, \mathbb{C})$$

G torsors over a complex variety X

$\{X_\alpha \to X\}$ $\qquad$ ⓪ $\tilde{X} = \bigcup_\alpha U_\alpha \to X$

$$\tilde{X} \underset{\text{étale}}{\twoheadrightarrow} X \xleftarrow{G} \mathcal{E}$$

$\mathcal{E}_{\tilde{X}}$ is trivial $\qquad \mathcal{E}_{\tilde{X}} \underset{\iota}{\simeq} \tilde{X} \times G$

$$\tilde{\tilde{X}} = \tilde{X} \times_X \tilde{X}$$

$$\begin{tikzcd} \tilde{\tilde{X}} \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \arrow[dd, bend left=60, "q"] \arrow[dd, bend right=60, "q"'] \\ \tilde{X} \arrow[d, "f"] \\ X \end{tikzcd} \qquad q^*\mathcal{E} = \Big\| \quad \begin{tikzcd} p_1^*\mathcal{E}_{\tilde{X}} \arrow[r, phantom, "\underset{p_1^*\iota}{\simeq}"] & \tilde{\tilde{X}} \times G \arrow[d, "g"] \\ p_2^*\mathcal{E}_{\tilde{X}} \arrow[r, phantom, "\underset{p_2^*\iota}{\simeq}"] & \tilde{\tilde{X}} \times G \end{tikzcd}$$

$$g \in Aut(\tilde{\tilde{X}} \times G) = Mor(\tilde{\tilde{X}} \to G)$$

Cocycle cond on triple product

$$Z(\tilde{X} \to X, G) \subset Mor(\tilde{\tilde{X}} \to G)$$

equivalence condition

$$\begin{tikzcd} \tilde{X} \times_X \tilde{X} \times_X \tilde{X} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\ \tilde{X} \times_X \tilde{X} \end{tikzcd} \qquad g_{12}, g_{23}, g_{13} \qquad g_{12} \circ g_{23} = g_{13}$$

Given $g \in Z(\tilde{\tilde{X}}, G)$

$$\begin{tikzcd} G \times \tilde{X} \arrow[r] \arrow[d, dashed] & \tilde{X} \arrow[d] \\ E \arrow[r] & X \end{tikzcd}$$

scheme

To descend need $p_1^*(G \times \tilde{X}) \simeq p_2^*(G \times \tilde{X})$

$$\begin{tikzcd} p_1^*(G \times \tilde{X}) \arrow[d, equal] & p_2^*(G \times \tilde{X}) \arrow[d, equal] \\ G \times \tilde{\tilde{X}} \arrow[r, "g"'] & G \times \tilde{\tilde{X}} \end{tikzcd}$$

$h \in Mor(\tilde{X}, G)$

$g \sim (p_1^*)^{-1} h \, g \, p_2^* h$ equivalence relation on $Z(\tilde{X} \to X, G)$

$Mor(\tilde{X}, G) \curvearrowright Z(\tilde{X} \to X, G)$

$$H^1_{\acute{e}t}(\tilde{X} \to X, G) := Z(\tilde{X} \to X, G) / Mor(\tilde{X}, G)$$

Thm. Iso classes of G-torsors over X that trivialize on $\tilde{X}$ are in bijection with

$$H^1_{\acute{e}t}(\tilde{X} \to X, G)$$

Ex. Complete the proof.

$$\hat{X} \twoheadrightarrow \tilde{X} \twoheadrightarrow G$$

$$H^1_{et}(\tilde{X} \to X, G) \hookrightarrow H^1_{et}(\hat{X} \to X, G)$$

$$\underset{\substack{\| \\ \text{iso classes of} \\ G\text{-torsors}}}{H^1_{et}(X, G)} = \varinjlim_{\tilde{X} \to X} H^1_{et}(\tilde{X} \to X, G)$$

Thm. $\exists$ a natural bijection between

$GL(r)$ - torsors $/ X$

v.b. of rank r

locally free sheaves of rank r

Pf. $\mathrm{LocalFree}_r(X) \leftarrow \mathrm{Bun}_r(X) \to \mathrm{Bun}_{GL(r)}(X)$

$\{U \mapsto \Gamma(U, E)\} \longleftarrow\!\shortmid\ E \longmapsto \{(\cancel{x}, e_1, e_2, \dots, e_r)$ in the fiber$\}$ $\boxtimes$

Claim. $\mathrm{LocalFree}_r(X)/\text{iso} \xrightarrow{\sim} H^1_{et}(X, GL(r))$

Pf. U complex variety

Ⓐ $\mathrm{Aut}(\mathcal{O}_U^{\oplus r}) = \mathrm{Mor}(U, GL(r))$

$\mathcal{O}_U^r \mapsto \mathcal{O}_U^r$ $\quad$ aut: $U \to GL_r(\mathbb{C})$

$\Big\updownarrow$ $\qquad\qquad\qquad\qquad\quad$ $\searrow \ \cap$ open

(a_{ij}) $\quad a_{ij} \in \mathcal{O}_U$ $\qquad$ $\mathrm{Mat}_{r \times r}(\mathbb{C})$

$A: U \to \mathrm{Mat}_{r\times r}(\mathbb{C})$

(B) LocFree sheaves are étale locally trivial

(C) $\exists$ étale descent for locally free sheaves

$\tilde{X} \xrightarrow{f} X$ is étale

$F \in QCoh(X)$

f^*F is locally free $\Leftrightarrow$ F is locally free ∎

Cor. Every $GL(r)$-torsor is Zar-locally trivial.

Pf. WTS $H^1_{Zar}(X, G) = H^1_{et}(X, G)$ for $G = GL(r)$

$H^1(\mathrm{Spec}\,\mathbb{C}, G) = \star$ $\qquad$ $H^1(\mathbb{A}^1_{\mathbb{C}}, G) = \star$

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\psi"'] & \mathrm{Spec}\,\mathbb{C} \arrow[dl] \\ x & \end{tikzcd}$$

Raghunathan-Ramanathan Thm

$\mathbb{P}^1_{\mathbb{C}}$ $\qquad$ algebraic torus $/\mathbb{C}$ is a group $\simeq (G_m)^r$

$G_m = \mathbb{C}^* = GL_1(\mathbb{C})$

$T \subset G$ maximal torus

if T is a torus and $\nexists$ subtorus in G with larger rank

Thm. $H^1_{et}(\mathbb{P}^1_{\mathbb{C}}, T) \twoheadrightarrow H^1_{et}(\mathbb{P}^1_{\mathbb{C}}, G)$ $\quad H \to G$ homo-

$Bun_H(X) \to Bun_G(X)$

$H^1_{et}(X, H) \to H^1_{et}(X, G)$

T - max torus

$$\begin{aligned}
&\parallel \\
&\prod_1^r H^1_{et}(\mathbb{P}^1_{\mathbb{C}}, \mathbb{G}_m) \\
&\parallel \\
&\prod_1^r Pic(\mathbb{P}^1) \\
&\parallel \\
&\mathbb{Z}^r \curvearrowleft W_{G,T}
\end{aligned}$$

Weyl group of G and T

$$N_G(T)/Z_G(T) = W_{G,T}$$

$$H^1_{et}(\mathbb{P}^1_{\mathbb{C}}, G) \approx \mathbb{Z}^r / W_{G,T}$$

$$T = \begin{pmatrix} * & & 0 \\ & * & \\ 0 & & \ddots \\ & & & * \end{pmatrix} \subset G(V)$$

$$\text{T-torsor } \overset{W}{\curvearrowright}(l_1, \dots, l_r) \mapsto l_1 \oplus \cdots \oplus l_r$$

$$\forall E \xrightarrow{\mathbb{A}^r} \mathbb{P}^1 \quad \exists l_i \quad E = l_1 \oplus \cdots \oplus l_r$$

Thm. $\mathcal{E} \xrightarrow{G} X$ (smooth connected) $\quad \exists U \underset{\text{open}}{\subset} X \quad U \neq \emptyset \quad \underset{\text{closed}}{Z = X - U}$

$\mathcal{E}|_U$ is trivial

then $\mathcal{E}$ is Zariski locally trivial.

$G \rightsquigarrow$ ~~group~~ reductive group scheme

$X \rightsquigarrow$ regular scheme

Thm. A. Notations as in Thm.

Let $x \in X$, then $\exists$

(i) open $V \subset X$, $x \in V$

(ii) $S \underset{\text{open}}{\subset} \mathbb{A}^{\dim X - 1}$

(iii) $V \xrightarrow{p} \mathbb{A}^1_S$ is étale

(iv) $F \to \mathbb{A}^1_S$ s.t. $p^* F \approx \mathcal{E}$

(v) A closed $Y \subset \mathbb{A}^1_S$ finite$/S$

s.t. $F|_{\mathbb{A}^1_S - Y}$ is trivial.

$$\begin{tikzcd} & \mathcal{E}|_V \arrow[r, dashed] & F \arrow[d] & \\ X \arrow[r, phantom, "\supset"] & V \arrow[r] & \mathbb{A}^1_S \arrow[d] \arrow[r, phantom, "\supset"] & Y \arrow[dl, "\text{finite}"] \\ & & S & \end{tikzcd}$$

Thm. B. S - complex variety

$F \xrightarrow{G} \mathbb{A}^1_S$

$\exists\, Y \subset \mathbb{A}^1_S$ s.t. Y is S-finite $F|_{\mathbb{A}^1_S - Y}$ trivial

$\Rightarrow \forall s \in S \;\; \exists\, s \in S' \underset{\text{open}}{\subset} S$ ~~trivial~~ $F|_{\mathbb{A}^1_{S'}}$ trivial

By Thm. B $\mathcal{F}|_{\mathbb{A}^2_{S'}}$ trivial

$\mathcal{E}|_V = p^*\mathcal{F}$ is trivial over $p^{-1}(\mathbb{A}^2_{S'})$

$\mathcal{E}$ is trivial over $\underset{\substack{\cup \\ x}}{p^{-1}(\mathbb{A}^2_{S'})}$

$G \subset GL_{\mathbb{C}}(V)$ $\quad X$ complex var.

$$\mathcal{E} \xrightarrow{G} \mathbb{A}^1_{\mathbb{C}} \times X = \mathbb{A}^1_X$$

$\exists\ Z \underset{\text{closed}}{\subset} \mathbb{A}^1_X$ and finite $/X$

$\mathcal{E}|_{\mathbb{A}^1_X - Z}$ is trivial

$\Rightarrow \forall x \in X \quad \exists$ Zariski open nbd $x \in U \subset X$

$\mathcal{E}|_{\mathbb{A}^1_{\mathbb{C}} \times U}$ is trivial

Thm.

$X = pt = \operatorname{Spec} \mathbb{C}$

$\mathcal{E} \to \mathbb{A}^1_{\mathbb{C}}$ trivial over non-empty open of $\mathbb{A}^1_{\mathbb{C}}$

$\Rightarrow \mathcal{E}$ is trivial

Functor of points

Aff Sch $\cong$ Rings^{op}

$$\begin{tikzcd} Sch \arrow[r, hook] \arrow[dr, bend right] & \operatorname{Funct}(Sch^{op}, Sets) \arrow[d] \\ & \operatorname{Funct}(Rings, Sets) \end{tikzcd}$$

$X \mapsto h_X$ $\quad h_X(Y) = \operatorname{Mor}(Y, X)$

$X \mapsto \operatorname{Mor}(\operatorname{Spec} R, X) = X(R)$

also fully faithful

Pf. $\quad \operatorname{Mor}(X_1, X_2) \to \operatorname{Mor}(\underline{X_1}, \underline{X_2})$

WTS bijection $\qquad F$

$\forall R$ $\quad$ $R \to R'$

$$\begin{tikzcd} X_1(R) \arrow[r] \arrow[d] & X_2(R) \arrow[d] \\ X_1(R') \arrow[r] & X_2(R') \end{tikzcd}$$

$X_1 = \bigcup U_i$ — open affines

$$\begin{tikzcd} \operatorname{Spec} R_i \arrow[r, "\sim"] \arrow[rr, bend right] & U_i \arrow[r, hook] & X_1 \end{tikzcd}$$

$\varphi_i \in X_1(R_i) \rightsquigarrow \varphi_i' \in X_2(R_i)$

$$\begin{tikzcd} \operatorname{Spec} R_i \arrow[r, "\varphi_i'"] & X_2 \\ U_i \arrow[u, "\wr"] \arrow[ur, "\varphi_i''"'] & \end{tikzcd}$$

check $\quad \varphi_i''|_{U_i \cap U_j} = \varphi_j''|_{U_i \cap U_j}$

$\Rightarrow \varphi'' : X_1 \to X_2$

Thus it is surjective $\quad \varphi'' \longmapsto F$ $\quad \blacksquare$

e.g.

$Sch_A \hookrightarrow Funct(A\text{-alg}, Sets)$

$Sch_{\mathbb{C}} \hookrightarrow Funct(\mathbb{C}\text{-alg}, Sets)$

$$\begin{tikzcd} Sch_{\mathbb{C}} \arrow[r, hook] & Funct(\mathbb{C}\text{-alg}, Sets) \\ Var_{\mathbb{C}} \arrow[u, phantom, "\cup"] \arrow[ur, hook] & \\ X \arrow[uur, mapsto] & \underline{X} \end{tikzcd}$$

$\underline{X}(\mathbb{C}) =$ closed pts of X

$\underline{X}(\mathbb{C}[\varepsilon]) =$ tangent vectors $\quad \mathbb{C}[\varepsilon] = \mathbb{C}[\varepsilon]/\varepsilon^2$

$$\begin{array}{c} \mathrm{Sch}_{\mathbb{C}} \\ \downarrow \\ \mathrm{Sh}_{fppf}(\mathbb{C}\text{-alg}, \mathrm{Sets}) \end{array}$$

G is a group

$\mathcal{X} \in \mathrm{Funct}(\mathbb{C}\text{-alg}, \mathrm{Sets})$

Action of G on X

$\forall$ $\mathbb{C}$-alg R action

$$G(R) \times \mathcal{X}(R) \to \mathcal{X}(R)$$

compatible with pullbacks $R \to R'$:

$$\begin{array}{ccc} G(R) \times \mathcal{X}(R) & \to & \mathcal{X}(R) \\ & & \downarrow \\ G(R') \times \mathcal{X}(R') & \to & \mathcal{X}(R') \end{array}$$

Ex. $\mathcal{X} = \underline{X}$, X is a scheme $\Leftrightarrow$ usual def.

Quotients

$$\mathcal{X}/G \in \mathrm{Funct}(\mathbb{C}\text{-alg}, \mathrm{Sets})$$

$$(\mathcal{X}/G)(R) = \mathcal{X}(R)/G(R)$$

If $\mathcal{X}$ is a scheme $(\mathcal{X} = \underline{X})$ (assume that the action is free)

Is $\mathcal{X}/G$ a scheme?

$G \circlearrowright X$

$X/G \Leftarrow$ it's hard to give a def, and that's why we want to use functor of points

Prop. $\mathcal{E} \xrightarrow{G} X \Leftarrow$ right torsor

$G \circlearrowright Y$, associated space

$\mathcal{E} \times^G Y = (\mathcal{E} \times Y)/G$

$g \cdot (e, y) = (eg^{-1}, gy)$

If Y is affine, then $\mathcal{E} \times^G Y$ is a scheme

~~$Y' \to Y$ étale~~

Pf. $X' \to X$ étale $\quad \mathcal{E}_{X'} := \mathcal{E} \times_X X' \simeq G \times_X X'$

$(\mathcal{E} \overset{G}{\times} Y) \times_X X'$

~~$X' \to X$~~ $\quad$ ~~Funct(Alg-$\mathbb{C}$, X) → Funct(Alg-$\mathbb{C}$, X)~~

~~X : Alg-$\mathbb{C}$~~ $\quad \mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathrm{Funct}(\mathbb{C}\text{-Alg}, \mathrm{Sets})$

$$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})(R) = \mathcal{X}(R) \times_{\mathcal{Y}(R)} \mathcal{Z}(R)$$

$$\begin{array}{ccc} X' \times Y & \to & \mathcal{E} \times^G Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

Now $(\mathcal{E} \times^G Y) \times_X X' = \mathcal{E}_{X'} \overset{G}{\times} Y = (X' \times G) \overset{G}{\times} Y = X' \times Y$

use étale descent for affine sch

$\mathcal{X} \in \mathrm{Funct}(\mathbb{C}\text{-alg}, \mathrm{Sets})$

$$\begin{tikzcd} \mathcal{X} \times_X X' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ X' \arrow[r, "\text{fpqc}"'] & X \end{tikzcd}$$

Prop. If $\mathcal{X} \times_X X'$ is represented by an affine scheme, then $\mathcal{X}$ is affine X-scheme. $\boxtimes$

$G \xrightarrow[\text{hom}]{\varphi} H$ $\quad G \curvearrowright H$ action

$g \cdot h = \varphi(g) h$

$$\begin{tikzcd} \mathcal{E} \arrow[d, "G"'] & \mathcal{E} \times^G H \curvearrowleft H \\ X & H\text{- torsor} \arrow[u] \end{tikzcd}$$

$\varphi_* :$ G - torsors $\to$ H - torsors

$H^1_{et}(X, G) \to H^1_{et}(X, H)$

Reduction of torsors

$H \overset{\varphi}{\hookrightarrow} G$ $\quad$ $\begin{matrix} \mathcal{E} \\ G \downarrow \\ X \end{matrix}$

An H-reduction of $\mathcal{E}$ is a pair $(\mathcal{E}_H, \iota)$

$$\begin{tikzcd} \mathcal{E}_H \arrow[d, "H"'] \\ X \end{tikzcd} \qquad \varphi_* \mathcal{E}_H \xrightarrow[\iota]{\simeq} \mathcal{E}$$

e.g.
$\{e\} \subsetneq G$

$$X \xrightarrow{e} X$$

$$\iota(\text{triv}) = X \times^{\{e\}} G = X \times G/\{e\} = X \times G$$
$$\begin{matrix} \cong \\ \mathcal{E} \end{matrix}$$

reduction $\rightarrow$ "trivilization"

$\mathcal{E} \xrightarrow{G} X \qquad H \subset G$

$\{H\text{-reductions of } \mathcal{E}\} = \{\text{sections } \mathcal{E}/H\}$

$$\begin{tikzcd} \mathcal{E}/H \arrow[d] \\ X \arrow[u, bend left, dashed] \end{tikzcd} \qquad (\mathcal{E}/H)(R) = \mathcal{E}(R)/H(R)$$

$\mathcal{E} \xrightarrow{G} \mathbb{A}^1_{\mathbb{C}} \qquad T \subset B \subset G$
$$T \cong (\mathbb{G}_m)^r$$

$$T = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_n = B \qquad B_i / B_{i-1} \simeq \mathbb{G}_a$$

G/B is a projective variety $\quad \mathcal{E}/B \underset{G/B}{\to} X$

$\mathcal{E}|_{\mathcal{U}} \subset \mathcal{U} \times G$

$\mathcal{U} \subset \mathbb{C}$

$\mathcal{E}|_{\mathcal{U}} \xrightarrow{\sim} \mathcal{U} \times G$

$$\begin{tikzcd} \mathcal{E}|_{\mathcal{U}}/B \arrow[r, "\sim"] & \mathcal{U} \times G/B \\ & \mathcal{U} \arrow[ul] \arrow[u, "u \mapsto (u, eB)"'] \end{tikzcd}$$

extends to $\mathbb{A}^1_{\mathbb{C}}$

$\mathcal{E} \times^G Y \underset{Y}{\to} X$

Prop. $\mathcal{X} \to X \leftarrow$ normal $\overset{\text{(connected)}}{\text{curve}}$

"projective" : $\exists X' \underset{\text{fpqc}}{\to} X$

$\mathcal{X} \times_X X'$ is a scheme, proj over X'

Then $\mathcal{U}$ open $\subset X$ $\quad s : \mathcal{U} \to \mathcal{X}$ extends to X

$$\begin{array}{c} \mathcal{E}_B \\ \downarrow \\ \mathbb{A}^1_{\mathbb{C}} \end{array} \quad \varphi^*_{\mathcal{X}} \mathcal{E}_B \simeq \mathcal{E}$$

enough to show $\mathcal{E}_B$ is trivial

$$H \underset{\text{normal}}{\hookrightarrow} G \to G/H$$

$$H^1_{et}(X, H) \to H^1_{et}(X, G) \to H^1_{et}(X, G/H)$$

$$H^1_{et}(X, G_a) = H^1_{et}(X, \mathcal{O}_X) = H^1_{Zar}(\underset{\uparrow \, \mathbb{A}^1_{\mathbb{C}}}{X}, \mathcal{O}_X) = 0$$

$$H^1(\mathbb{A}^1_{\mathbb{C}}, G_m) = \mathrm{Pic}(G_m) =$$ [illegible]

$\mathcal{E} \xrightarrow{G} \mathbb{A}^1_{\mathbb{C}}$ trivial away from finite $\Rightarrow \mathcal{E}$ is trivial

Pf. $T \subset B \subset G$

$T \cong (\mathbb{G}_m)^r$

G/B projective $\quad T = B_0 \subset B_1 \subset \cdots \subset B_N = B$

$\forall\, B_i/B_{i-1} \cong \mathbb{G}_a \leftrightarrow$ "$\mathbb{C}$"

Ex. $G = GL_n$

$$T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \qquad B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$$ upper triangular

$$T \subset \begin{pmatrix} * & & 0 & * \\ & \ddots & & 0 \\ 0 & & & * \end{pmatrix} \subset \begin{pmatrix} * & & 0 & * \\ & \ddots & & * \\ & & & 0 \\ 0 & & & * \end{pmatrix} \subset \cdots$$

$G/B = \{ F_0 = \{0\} \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = \mathbb{C}^n \}$

$\mathrm{Tors}(\mathbb{A}^1_{\mathbb{C}}, T) \to \mathrm{Tors}(\mathbb{A}^1_{\mathbb{C}}, B) \twoheadrightarrow \mathrm{Tors}(\mathbb{A}^1_{\mathbb{C}}, G)$

$$(\mathcal{E}_B \xrightarrow[B]{} \mathbb{A}^1_{\mathbb{C}}) \mapsto (\mathcal{E}_B \overset{B}{\times} G \xrightarrow[G]{} \mathbb{A}^1_{\mathbb{C}})$$

Find the inverse:

B-reduction $\to \{ \mathcal{E}_B \xrightarrow{B} \mathbb{A}^1_{\mathbb{C}}, \quad \mathcal{E} \xrightarrow{G} \mathbb{A}^1_{\mathbb{C}}$

$\mathcal{E}_B \overset{B}{\times} G \xrightarrow[\sim]{} \mathcal{E} \}$

$\quad ||$

$\{\text{sections of } \mathcal{E}/B \to \mathbb{A}^1_{\mathbb{C}}\}$

$$\begin{array}{ccc} B & \to & \mathcal{E}(R)/B(R) \\ \uparrow & & \uparrow \\ \mathbb{C}\text{-alg} & & \mathcal{E}/B := \text{sheafification in étale topology} \end{array}$$

$\mathcal{E}/B \to X \qquad \mathcal{E} \xrightarrow{G} X$, $s : X \dashrightarrow \mathcal{E}$

Want: $F \xrightarrow[B]{} X$

$$F_x = \{ e \in \mathcal{E}_x,\ e \bmod B = s(x) \}$$

$$F = \bigcup F_x \subset \mathcal{E}$$

May work étale locally, may assume $\mathcal{E}$ is trivial

$X \underset{\text{open}}{\subset} \mathbb{A}^1_{\mathbb{C}} \qquad \mathcal{E}|_X \approx G \times X$

$$\begin{tikzcd} X \arrow[r] \arrow[rrr, bend right, "s"'] & X \times G \arrow[r] & X \times G/B = \mathcal{E}_X/B & = (\mathcal{E}/B)_X \end{tikzcd}$$

can extend to $\mathbb{A}^1_{\mathbb{C}} \to \mathcal{E}/B$

(valuative criterion of properness)

long exact sequence $\quad \mathrm{Tors}(\mathbb{A}^1_{\mathbb{C}}, T) \to \mathrm{Tors}(\mathbb{A}^1_{\mathbb{C}}, B) \twoheadrightarrow \mathrm{Tors}(\mathbb{A}^1_{\mathbb{C}}, G)$

$$+ \quad H^1(\mathbb{A}^1_{\mathbb{C}}, G_a) = 0$$
$$\qquad\ \, \|$$
$$H^1(\mathbb{A}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{A}^1}) = 0$$

$$\Rightarrow \mathrm{Tors}(\mathbb{A}^1_{\mathbb{C}}, T) = H^1(\mathbb{A}^1_{\mathbb{C}}, G_m) = 0$$

$\mathrm{Tors}(X, B) \twoheadrightarrow \mathrm{Tors}(X, G)$ for any Dedekind schemes (e.g. any smooth curve)

$$\mathrm{Tors}(X, T) \twoheadrightarrow \mathrm{Tors}(X, B)$$

$$\uparrow$$

$$H^1(X, \mathcal{O}_X) = 0 \quad (\text{e.g. } X = \mathbb{P}^n_k)$$

Torsors on $\mathbb{P}^1$

$\hat{\mathcal{E}} \to \mathbb{P}^1_{\mathbb{C}}$ trivial over open

$$\hat{\mathcal{E}}|_{\mathbb{A}^1_{\mathbb{C}}} \underset{\sigma_1}{\simeq} G \times \mathbb{A}^1_{\mathbb{C}}$$

$$\hat{\mathcal{E}}|_{\mathbb{P}^1 - 0} \underset{\sigma_2}{\simeq} G \times (\mathbb{P}^1 - 0)$$

$\mathbb{A}^\times_{\mathbb{C}} = \mathbb{A}^1_{\mathbb{C}} - 0$

$$\begin{array}{ccc} \hat{\mathcal{E}}|_{\mathbb{A}^\times_{\mathbb{C}}} & \underset{\sigma_1^*}{\simeq} & G \times \mathbb{A}^1_{\mathbb{C}} \\ \| & & \downarrow \tau \\ \hat{\mathcal{E}}|_{\mathbb{A}^\times_{\mathbb{C}}} & \underset{\sigma_2^*}{\simeq} & G \times \mathbb{A}^1_{\mathbb{C}} \end{array}$$

$$T : \mathbb{A}^\times_{\mathbb{C}} \to G$$

$$\uparrow$$

$$G(\mathbb{C}[t^{-1}, t])$$

$D = \mathrm{Spec}\, \mathbb{C}[[t]] = \{\cdot, \star\}$ ($\neq$ formal scheme

$\mathring{D} = \mathrm{Spec}\, \mathbb{C}((t))$ $\quad\neq \varinjlim \mathrm{Spec}\, \mathbb{C}[t]/t^n$)

$$\|$$

$$\mathbb{C}[[t]]_t$$

$D \subset \mathbb{A}^1 = \mathbb{P}^1 - 0 \hookrightarrow \mathbb{P}^1$

$\mathbb{C}[t] \hookrightarrow \mathbb{C}[[t]]$

$\mathbb{P}^1 = \mathbb{A}^1 \sqcup_{\mathring{D}} D$ fpqc cover

$T \in G((t)) = G(\mathbb{C}((t)))$

$\downarrow$

$\text{glue}(T) := G \times \mathbb{A}^1_{\mathbb{C}} \sqcup_{T, \mathring{D}} G \times D$

$\mathbb{C}((t)) = \mathbb{C}[[t]]_t = \{ a_n t^{-n} + \cdots + a_1 t^{-1} + b_0 + b_1 t + \cdots + b_n t^n + \cdots \}$

$\mathcal{E} \xrightarrow{G} \mathbb{A}^1_R$ $\qquad$ $\mathbb{A}^1_R = \mathbb{A}^1 \times \operatorname{Spec} R$

$\mathcal{E}|_{\mathbb{A}^1_R - Y}$ is trivial

Y is finite over R

Then $x \in \operatorname{Spec} R$ $\exists$ open $x \in U \subset \operatorname{Spec} R$

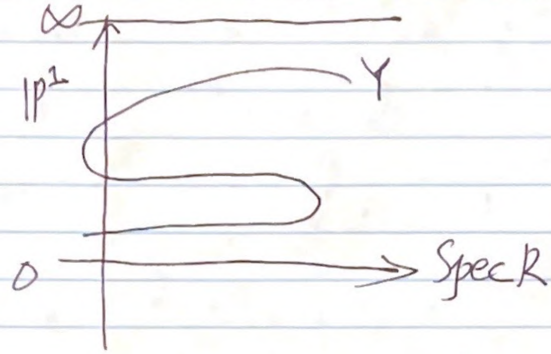
$\mathcal{E}|_{\mathbb{A}^1_U}$ is trivial

Y is finite over R

(i) $\forall x \in \operatorname{Spec} R$ Y_x is finite

(ii) $Y \hookrightarrow \mathbb{A}^1_R \hookrightarrow \mathbb{P}^1_R$ $\qquad Y \cap \infty = \phi$

closed (because it is finite)



Step 1. Extend $\mathcal{E}$ to $\hat{\mathcal{E}} \xrightarrow[G]{} \mathbb{P}^1_R$

$$\mathbb{P}^1_R = \underset{\substack{\uparrow \\ \mathcal{E}}}{\mathbb{A}^1_R} \bigsqcup_{\mathbb{A}^1_R - Y} \underset{\substack{\uparrow \\ \mathcal{E}_{triv}}}{\mathbb{P}^1_R - Y} \leftarrow \text{open b/c } Y \text{ is } R\text{-finite}$$

Step 2. May assume $\underset{\hat{\mathcal{E}}_x}{\hat{\mathcal{E}}|_{\mathbb{P}^1_x}}$ is trivial

$\hat{\mathcal{E}}_x|_{\mathbb{A}^1_x}$ is trivial

$\hat{\mathcal{E}}_x|_D$ is trivial

$\hat{\mathcal{E}}_x = \text{Glue}(\tau) \qquad \tau \in G(\mathbb{C}((t)))$

Lemma. $G(\mathbb{C}((t))) = \underset{\gamma^+}{G(\mathbb{C}[[t]])} B(\mathbb{C}((t)))$

Pf. γ:
$$\begin{tikzcd} \mathring{D} \arrow[r] & G \arrow[r] & G/B \\ D \arrow[u, phantom, "\cap"] \arrow[ur, "\gamma_+"] \arrow[urr, "\tilde{\gamma}"'] & & \end{tikzcd}$$

$\tau T_+^{-1} \in B(\mathbb{C}((t)))$

$\boxtimes$

· May assume $\tau \in B(\mathbb{C}((t)))$

$$\begin{array}{c} D \sqcup \mathbb{P}^2_{\mathbb{C}} - \infty \\ \uparrow \\ \tau_x \in \mathrm{Aut}(G \times \mathbb{C}[[t]]) \end{array}$$

$\mathbb{A}^1_R = \mathrm{Spec}\, R[[t]]$

$x - m \subset R$

$R((t))/m[[t]] \simeq \mathbb{C}((t))$

· Claim. $B(R((t))) \twoheadrightarrow B(\mathbb{C}((t)))$ surjective

$$B = T \times U, \quad T = (G_m)^r$$

$$\begin{cases} G_m(R((t))) \twoheadrightarrow G_m(\mathbb{C}((t))) \\ U(R((t))) \twoheadrightarrow U(\mathbb{C}((t))) \end{cases}$$

For G_m: $\quad R((t))^\times \to \mathbb{C}((t))^\times \qquad R \twoheadrightarrow R/m = \mathbb{C}$

$$\{t^n(\tilde{a}_0 + \tilde{a}_1 t + \cdots)\} \qquad \mathbb{C}((t))^\times = \{t^n(a_0 + a_1 t + \cdots + a_n t^n + \cdots)\}$$

$a_i \in \mathbb{C} \rightsquigarrow \tilde{a}_i \in R$

For $U \simeq \mathbb{C}^N$: $\quad \prod_1^N R((t)) \to \prod_1^N \mathbb{C}((t))$

$$E \xrightarrow{G^N} X$$

$$\mathrm{Sect}(X, E) \to \mathrm{Sect}(Y, E)$$

$$Y \subset X$$

τ lifts $\tilde{\tau} \in B(R((t))) \subset G(R((t)))$

$$\hat{\mathcal{E}} = \mathcal{E} \bigsqcup_{\mathbb{P}^1_R - Y} \mathcal{E}_{triv}$$

$D_R = \mathrm{Spec}\, R[[t]]$ $\qquad$ $\mathbb{P}^1_R = \mathbb{A}^1_R \bigsqcup_{\mathring{D}_R} D_R$

$$\mathring{D}_R = \mathrm{Spec}\, R((t)) = \mathrm{Spec}\, R[[t]]_t$$

$$\begin{tikzcd} \mathring{D}_R \arrow[r, hook] \arrow[d] & \mathbb{P}^1_R \\ \mathbb{P}^1_R - Y \arrow[ur] & \end{tikzcd} \Rightarrow \mathcal{E}|_{\mathring{D}_R} \text{ is trivial}$$

$\hat{\hat{\mathcal{E}}} := \{\mathcal{E}|_{\mathring{D}_R}$ glue with trivial over D_R using $\tilde{\tau}^{-1}\}$

$\hat{\hat{\mathcal{E}}}|_{\mathbb{P}^1_x}$ is trivial

^ is obtained from $\hat{\mathcal{E}}|_{\mathbb{P}^1_x}$ by τ^{-1}

$G \subset GL_{n,\mathbb{C}}$

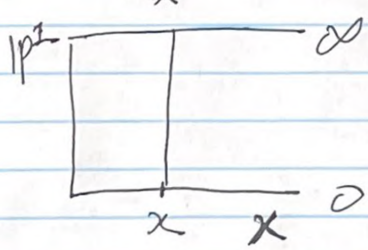
Thm. $\mathcal{E} \xrightarrow{G} \mathbb{A}^1_{\mathbb{C}} \times X$

$\exists\ Y \subset \mathbb{A}^1 \times X$ s.t. Y is X-finite and $\mathcal{E}|_{\mathbb{A}^1 \times X - Y}$ is trivial, then $\forall x \in X$ $\exists$ Zariski nbd U s.t. $\mathcal{E}|_{\mathbb{A}^1 \times U}$ trivial.

$Y \underset{\text{closed}}{\subset} \mathbb{P}^1 \times X$ extend $\mathcal{E}$ to $\tilde{\mathcal{E}}$ on $\mathbb{P}^1 \times X$

(gluing with trivial over $\mathbb{P}^1 \times X - Y$)

$\mathcal{E}|_{\mathbb{A}^1_x}$ is trivial ($\mathbb{A}^1_x \sim \mathbb{A}^1_{\mathbb{C}}$)



$\rightsquigarrow \alpha \in G((t)) \subset G[[t]]B((t)) = G(D)B(D)$

$= \alpha_+ \alpha_-$

$\tilde{\alpha}_- \in B(\mathring{D} \times X)$ $\quad \mathring{D} = \mathrm{Spec}\, \mathbb{C}((t))$

Use $\tilde{\alpha}_-$ to change $\tilde{\mathcal{E}}$ (but don't change $\mathcal{E}$)

then $\tilde{\mathcal{E}}|_{\mathbb{P}^1_x}$ is trivial

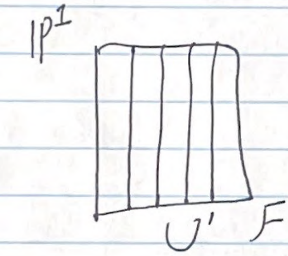
Prop. $\tilde{\mathcal{E}} \to \mathbb{P}^1 \times X$ — compact (projective)

$\tilde{\mathcal{E}}|_{\mathbb{P}^1 \times x}$ is trivial, then $\exists$ open nbd U' of x

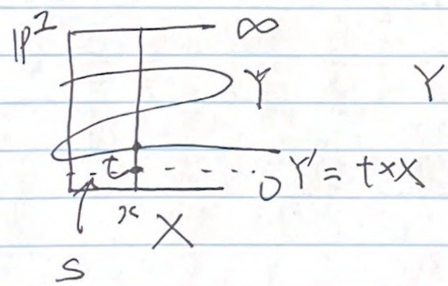
$F \xrightarrow{G} U'$, $\mathcal{E}|_{\mathbb{P}^1 \times U'} \simeq p_{U'}^* F$ $\quad p_U : \mathbb{P}^1 \times U' \to U'$

Rem. $u \in U'$ $\tilde{\mathcal{E}}|_{\mathbb{P}^1 \times u}$ is trivial

$\simeq p_u^* F_u \qquad p_u : \mathbb{P}^1_u \to u$



Proof of Thm. $t \in \mathbb{P}^1_x - Y$



$Y \cap Y' \xrightarrow{\text{finite}} X$

$p_x(Y \cap Y') \subset X$

$\not\ni$

x

$U := U' - p_x(Y \cap Y')$ open

$\ni$

x

Replace X with U

$\tilde{\mathcal{E}} = p_x^* F$, $Y \cap Y' = \phi \Rightarrow Y' \subset \mathbb{P}^1 - Y$

$\tilde{\mathcal{E}}|_{t \times X} = \tilde{\mathcal{E}}_{Y'}$ is trivial

$$s : X \to \mathbb{P}^1 \times X$$
$$x \mapsto (t, x)$$

$$\mathcal{F} = s^* p_X^* \mathcal{F} \qquad p_X \circ s = id_X$$
$$= s^* \tilde{\mathcal{E}}$$
$$= \tilde{\mathcal{E}}|_{t \times X} \quad \text{trivial}$$
$$\Rightarrow \quad \tilde{\mathcal{E}} = p_X^* \mathcal{F} \text{ is trivial}$$

$\blacksquare$

Y - proj. variety $\quad \mathcal{Y} := Bun_G(Y)$

$$\mathcal{Y} : Sch^{op} \to \text{Groupoid}$$
$$\mathcal{Y}(S) = \{ \mathcal{E} \xrightarrow{G} Y \times S \}$$

$$BG = Bun_G(pt) \qquad BG(S) = \{ \mathcal{E} \xrightarrow{G} S \}$$

$$\{ \tilde{\mathcal{E}} \to \mathbb{P}^1 \times X \} \in Bun(\mathbb{P}^1)(X)$$

$$\tilde{\mathcal{E}} : \quad X \xrightarrow{e} Bun(\mathbb{P}^1) \overset{open}{\supset} Bun^{trivial}(\mathbb{P}^1)$$

$$e|_x \subset Bun^{trivial}(\mathbb{P}^1)$$

$$x \in \mathcal{U} = e^{-1}(Bun^{trivial}(\mathbb{P}^1))$$
$$\underset{X}{\overset{\text{open}}{\cap}}$$

claim. $\quad Bun^{trivial}(\mathbb{P}^1) = BG$

$\mathbb{P}^1 \to \operatorname{Spec}\mathbb{C}$

$\mathbb{P}^1 \times S \to S$

$$\begin{array}{ccc} \mathrm{Bun}(\mathbb{P}^1) & \xleftarrow{\text{pullback}} & \mathrm{Bun}(\mathcal{X}) \\ & \text{open embedding} & \| \\ & & BG \end{array}$$

$$\begin{tikzcd} & e|_U & E|_U \\ & U \arrow[r] \arrow[d] & \mathrm{Bun}(\mathbb{P}^1) & p^*F \\ & BG \arrow[ur] & \end{tikzcd}$$

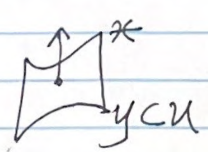
$F \xrightarrow{G} U$ WTS the open condition

Prop. $\mathcal{X}$ algebraic stack locally of finite type $/\mathbb{C}$

$\mathcal{Y} \subset \mathcal{X}$ locally closed reduced

$$\begin{tikzcd} \forall\ \operatorname{Spec}\mathbb{C}[\varepsilon]/\varepsilon^2 \arrow[r, "\varphi"] & \mathcal{X} \\ \operatorname{Spec}\mathbb{C} \arrow[u, hook] \arrow[r] & \mathcal{Y} \arrow[u, hook] \end{tikzcd}$$

then φ factors through $\mathcal{Y} \Rightarrow \mathcal{Y}$ open.



algebraic: $\exists\, X \xrightarrow{\text{smooth}} \mathcal{X}$

scheme

$\mathrm{Spec}\ \mathbb{C}[\varepsilon]/\varepsilon^2 \to \mathrm{Bun}(Y)$ $\qquad$ Y any projective variety

$\mathcal{E} \to Y_{\mathbb{C}[\varepsilon]/\varepsilon^2}$

$\mathcal{E}|_Y$ is trivial

$\downarrow$

$H^1(Y, \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_Y)$ $\qquad$ $\mathfrak{g} = \mathrm{Lie}(G)$

$\bigoplus_1^{\dim \mathfrak{g}} \mathcal{O}_Y$

$X = \bigcup_\alpha U_\alpha$

$X[\varepsilon] = \bigcup_\alpha U_\alpha[\varepsilon]$ $\qquad$ $X[\varepsilon] := X \times \mathrm{Spec}\ \mathbb{C}[\varepsilon]/\varepsilon^2$

$g_{\alpha\beta} \in \mathrm{Mor}(U_{\alpha\beta}[\varepsilon], G)$

$g_{\alpha\beta}|_{\varepsilon=0} = \text{identity}$

$\quad \| $

$Id + \varepsilon \xi_{\alpha\beta}$ $\qquad$ $\xi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Lie}(G)$

$(Id + \varepsilon \xi_{\alpha\beta})(Id + \varepsilon \xi_{\beta\gamma}) = Id + \varepsilon \xi_{\alpha\gamma}$

$\Rightarrow \xi_{\alpha\beta} + \xi_{\beta\gamma} = \xi_{\alpha\gamma}$

In our case $H^1(\mathbb{P}^1, \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}) \simeq \mathfrak{g} \otimes_{\mathbb{C}} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$

$= 0$

$\blacksquare$

$SL(n)$ $PGL(n)$ $\qquad$ $SL(n) \subset GL(n)$

$1 \to SL(n) \hookrightarrow GL(n) \xrightarrow{\det} \mathbb{G}_m \to 1$

$$\to H^0(X, \mathbb{G}_m) \to H^1(X, SL(n)) \to H^1(X, GL(n))$$

$= $ v.b.

$\to H^1(X, \mathbb{G}_m)$

$= $ l.b.

$V \to \Lambda^n V$ $\quad$ $n = $ top

$$H^1(X, SL(n)) = \{V, \Lambda^{top} V \cong X \times \mathbb{A}^1\}$$

$\mathbb{P}^1$ $\quad$ $n = 2$

$\mathcal{O}(n) \oplus \mathcal{O}(m)$ $\quad$ det trivial $\Leftrightarrow$ $n = -m$

$\mathcal{O}(n) \oplus \mathcal{O}(-n)$

$$\mathrm{Aut}(\mathcal{O} \oplus \mathcal{O}) = \mathrm{Maps}(\mathbb{P}^1, GL(2)) = GL(2)$$

$$\mathrm{Aut}(V \approx \mathcal{O} \oplus \mathcal{O}, \Lambda^2 V \cong \mathcal{O}) = SL(2)$$

$SL(n) \subset GL(n)$

semi-simple $\quad$ reductive

no normal $\mathbb{G}_a$ or $\mathbb{G}_m$ $\quad$ no normal $\mathbb{G}_a$

Rank 1. $\quad$ $GL(2)$ $\quad$ $SL(2)$ $\quad$ $PGL(2)$

$$1 \to \mathbb{G}_m \to GL(2) \to PGL(2) \to 1$$

$$H^1(X, \mathbb{G}_m) \to H^1(X, GL(2)) \to H^1(X, PGL(2)) \to H^2(X, \mathbb{G}_m)$$

$H^1(X, \mathbb{G}_m) = $ l.b. $\quad$ action $\quad$ $H^1(X, GL(2)) = $ v.b.

$H^2(X, \mathbb{G}_m) = $ Brauer group (X quasiproj. smooth) $\quad$ $= 0$ if X complex curve

$E \mapsto E \otimes \ell$

X curve $\Rightarrow H^1(X, PGL(2)) = H^1(X, GL(2))/H^1(X, \mathbb{G}_m)$

$$\{\text{v.b.}\} / \otimes \text{ with l.b.}$$

$F = \mathbb{C}$ $\quad G = PGL(\mathbb{C})$

$$\begin{tikzcd} & Z \arrow[dl] \arrow[d] \arrow[r] & X/S \\ Bun_G(X,S) & Bun_G(X,S) \arrow[d, dashed, "S"] & \\ & \mathbb{A}^n & \mathbb{P}^n \end{tikzcd}$$

$X = \mathbb{P}^1$ or ell. curve

S - finite subset of X

$x \in X \setminus S$

$$\begin{tikzcd} & Z_x \arrow[dl] \arrow[dr] & \\ Bun_G & & {}^0Bun_G \end{tikzcd}$$

$$\begin{tikzcd} & Z \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ Y_2 & & Y_1 \end{tikzcd}$$

$f : Y_1 \to Y_2 \quad Z = \Gamma(f_1) \subset Y_1 \times Y_2$

$f = \pi_2 \circ (\pi_1)^{-1}$

$Fun(Y_1) \quad Fun(Y_2) \quad f^* : Fun(Y_2) \to Fun(Y_1)$

$f^* = ((\pi_1)^{-1})^* \circ (\pi_2)^*$

$((\pi_1)^{-1})^* = (\pi_1)_*$

π_1 has finite fibers

$(\pi_1)_* =$ sum over fibers

$(f^* g)(y) = \sum_{z \in f(y)} g(z) \Leftarrow$ Hecke operator

Now $Y_1 = Y_2 = Bun_G(X/\mathbb{F}_q)$

$Z =$ Hecke correspondence for G, X

$L^2(Y_1) \qquad L^2(Y_2)$

$\cong$

$\ell^2 = \{(x_i) \mid \Sigma x_i < \infty\}$

$L^2(\mathrm{Bun}_G) = \bigoplus H_K \qquad \psi \in H_K$

$H_K \psi = \beta(x)\psi$

What if we take $\mathbb{C}$ instead of $\mathbb{F}_q$?

1) no natural measure on $\mathbb{C}(\mathrm{Bun}_G)$
2) π_* has infinite fibers

Take sheaves instead of functions

$$\begin{array}{ccc} & Z & \\ {}^{q_2}\swarrow & & \searrow^{q_1} \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \end{array}$$

$(q_2)_*(q_2)^*$ is defined

Geometric Langlands correspondence

☆ Idea: add pts are hard for infinite but add v.s. easy (just direct sum)

$\mathrm{Bun}_G(X/\mathbb{F}_q, S)$ pts are of different weight

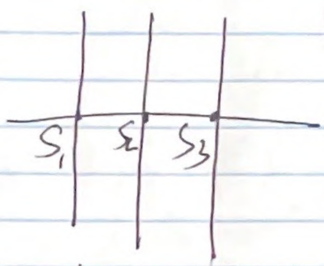
$w(E) = \frac{1}{|\#\mathrm{Aut}\,E|}$ "stacky"

$\mathrm{Bun}_G(X, S)$ G-bundles on X with parabolic structure at S

$PGL_2 \rightsquigarrow$ vector bundle of rk 2 on $X/E_{\mathbb{F}}$

if $F \simeq E \otimes \mathcal{L} \leftarrow$ l.b.

parabolic structure at $s \in S$ is choice of line in E_s



we choose for all $s \in S$

Example.

$G = PGL_2 \quad X = \mathbb{P}^1 \qquad Bun_G = \underset{\text{even deg}}{Bun_2} \sqcup \underset{\text{odd deg}}{Bun_1}$

$\det \simeq \mathcal{O}(K)$ ~~&~~ K even or odd using $k=0, 1$

$\mathcal{O}(K) \oplus \mathcal{O}(-K)$

~~Aut~~ ~~K=0~~ ~~zero~~ ~~PGL~~ $K>0$ $\qquad \mathcal{O}(-K) \to \mathcal{O}(K)$

$\mathcal{O} \to \mathcal{O}(2K)$

dim Aut $\quad 3,\ k=0 \quad PGL_2 \qquad 2k+1$

$2k+2,\ k>0$

open subset of Bun_0 is given by trivial bundle

$(\mathbb{P}_1)^S / PGL_2 \sim$ diagonal action on $(\mathbb{P}_1)^S$

$\downarrow S$

$(\mathbb{P}_1)^{|S|-3}$

$|S| \geq 4 \quad \mathbb{P}^1$

$|S| \geq 1 \quad$ ell. curve

any $n \quad$ genus ≥ 2

$\mathrm{Bun}_0 \simeq (\mathbb{P}^1)^{|S|-3}$

$X=$ ell. curve

Bun_1 has one bundle is iso choose in lines

$(\mathbb{P}_1)^{|S|}$

1- variety smooth $\underset{\cancel{|L|}}{\mathcal{L}}$ l.b. on $Y \overset{=\bigcup U_i}{}$ 1-gluing function

$g_{ij}: U_i \cap U_j \to \mathbb{C}^\times$

$z \mapsto |z|^c \downarrow$

$|\mathcal{L}|^c$ ~~gluing~~ $\mathbb{R}_{\geq 0}$ - bundle

$\mathbb{R}_{\geq 0}$

gluing functions $|g_{ij}|^{\mathbb{C}}$

$|\mathcal{L}|^2 = \mathcal{L} \otimes \bar{\mathcal{L}}$

$\mathcal{L} = K_Y$ bundle of densities $\Omega_Y \overset{=|K_Y|^2}{}$ f-section of Ω_Y

$\int_Y f$ is well-defined

$f \in C_c^\infty(U_i \cap U_j, \mathcal{L})$

$$f \rightsquigarrow \eta_i f \in C_c^\infty(U_i, \mathcal{L})$$

$$\downarrow$$

$$C_c^\infty(U_i)$$

$dy_1 \dots dy_k$ $\qquad$ $\sqrt{y_1}, \dots, \sqrt{y_k}$

$\Omega_Y^{\frac{1}{2}}$ $\qquad$ f, g we can define $(f,g) = \int_Y f\bar{g}$

$//$

$|K_Y|$ $\qquad$ $L^2(Y) =$ completion of compactly supported sections of $\Omega_\gamma^{\frac{1}{2}}$ - wrt $(\cdot,\cdot)$

Now we have a natural measure

$q_2^* f$ - section of $q_2^* \Omega_{Bun}^{\frac{1}{2}} \simeq q_1^* \Omega_{Bun}^{\frac{1}{2}} \otimes \Omega_1$

bundle of rel. densities along q_1

$(q_1)_*(q_2)^*$

integral over Ω_i piece

$$(H_x F)(y_1, \dots, y_m) = \int_{\mathbb{C}} F\left(\frac{t_1 - x}{s - y_1}, \dots, \frac{t_m - x}{s - y_m}\right) \frac{ds\,d\bar{s}}{\prod (s - y_i)^2}$$

change of coordinates from

Hecke modification

$S = \{t_0, t_1, \dots, t_m, t_{m+1} = \infty\}$

$(\mathbb{P}_2)^{m-2}$

$$L^2((\mathbb{P}^2)^{m-2}) \simeq L^2(\mathbb{C}^{m-2})$$

conjectures: H_x are defined and bounded

$\|H_x f\| \leq C(x) \|f\|$ in all cases

consider joint spectrum $\mathcal{H} = \oplus \mathcal{H}_k$ $H_x \psi = \beta_k(x) \psi$

$\psi \in \mathcal{H}_k$

1-to-1 between spectrum and

real opers $(\ )_x^2 + \sum \frac{1}{(x-t_i)^2} + \sum \frac{\mu_i}{(x-t_i)} + \beta_k = 0$

G

proved on $\mathbb{P}^1$ PGL_2 EHK

$\odot$ almost all PGL_2 EHK

$\mathbb{Q}(\varepsilon)/\mathbb{Q}(\varepsilon^k)$ almost $\mathbb{P}^1$ PGL_2 higher

Thm. $X/\mathbb{C}$ smooth $\quad x \in X \quad \underset{\text{closed}}{Z \subset X}$ hypersurface

Then $\exists$ (i) open nhd U of x

(ii) $S \underset{\text{open}}{\subset} \mathbb{A}^{\dim X - 1}_{\mathbb{C}}$

(iii) $U \xrightarrow{f} S \times \mathbb{A}^1$ s.t.

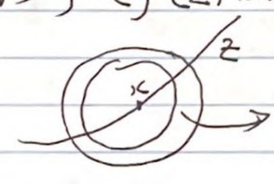
(i) $f|_{Z \cap U}$ is a closed embedding

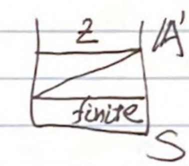
(ii) $Z \cap U \hookrightarrow U \xrightarrow{f} S \times \mathbb{A}^1 \Rightarrow S$ finite

(iii) f is étale

(iv) $f^{-1}(f(Z \cap U)) = Z \cap U$

$$\begin{tikzcd} x \in U \arrow[r] & S \times \mathbb{A}^1 \arrow[d] \\ U \cap Z \arrow[u, phantom, "\cup" description] \arrow[ur, dashed] \arrow[r, dashed, "\text{finite}"'] & S \end{tikzcd}$$





Elementary ~~fibrations~~ distinguished square Veovodsky

U – complex variety

$Z' \subset U$ closed $\qquad f|_{Z'}$ is a closed embedding

$$\begin{tikzcd} U \arrow[r, "\text{étale}"] & W \\ U - Z' \arrow[u, phantom, "\text{open} \cup" description] \arrow[r, "f'"] & W - Z'' \arrow[u, phantom, "\cup \text{ open}" description] \end{tikzcd}$$

(surjective)

cartesian, $f^{-1}(Z'') = Z'$

$Z'' := f(Z')$

Thm. G - complex linear group

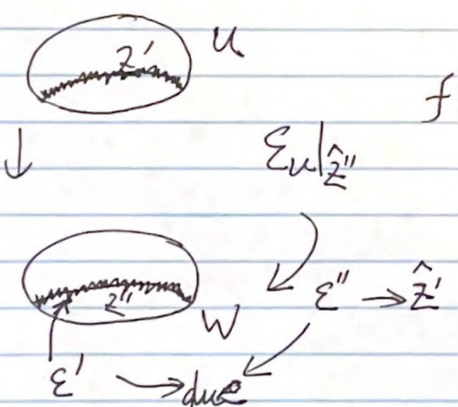
$\mathcal{E}_U \xrightarrow{G} U$

$\mathcal{E}' \xrightarrow{G} W - Z''$ compatible $\mathcal{E}_U|_{U-Z'} \simeq f'^{*}\mathcal{E}'$

$f' : U - Z' \to W - Z''$

Then $\exists\ \mathcal{E} \xrightarrow{G} W$

$$\begin{cases} f^{*}\mathcal{E} \overset{s}{\xrightarrow{\sim}} \mathcal{E}_U \\ \mathcal{E}|_{W-Z''} \overset{t}{\xrightarrow{\sim}} \mathcal{E}' \end{cases}$$



$f|_{\hat{Z}'} \xrightarrow{\sim} \hat{Z}''$ over the completion

Cor. $\mathcal{E}_U \xrightarrow{G} U$. $\mathcal{E}|_{U-Z'}$ is trivial $\Big\}$ Then $\exists\ \mathcal{E} \xrightarrow{G} W$ $f^{*}\mathcal{E} \simeq \mathcal{E}_U$

$\mathcal{E}|_{W-Z''}$ is trivial

~~$\mathcal{E}|_{U-Z'}$ is trivial~~

Remark. $U \sqcup W - Z'' \to W$ ~~Nisnevich~~ Nisnevich covers

↑ étale cover

$\mathcal{E} \xrightarrow{G} X \ni x$ $\qquad$ $\mathcal{U} \xrightarrow{f} W \hookrightarrow \mathbb{A}^1 \times S$

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{f} & W \\ \cup & & \cup \\ \mathcal{U} - Z' & \to & W - Z'' \end{array}$$

$\exists\, Z \underset{\text{closed}}{\subset} X$ $\qquad$ $\mathcal{E}|_{X-Z}$

$\mathcal{E}_{\mathcal{U}} := \mathcal{E}|_{\mathcal{U}}$ $\qquad$ $\mathcal{E}_{\mathcal{U}}|_{\mathcal{U} - Z \cap \mathcal{U}}$ is trivial

$W := f(\mathcal{U})$ $\qquad$ $Z' = Z \cap \mathcal{U}$

$\qquad\qquad\qquad\qquad$ $Z'' = f(Z')$

$\exists\, \mathcal{E}' \xrightarrow{G} W$ $\quad$ $f^*\mathcal{E}' \simeq \mathcal{E}_{\mathcal{U}}$ $\quad$ $\mathcal{E}'|_{W - Z''}$ trivial

claim. can extend $\mathcal{E}'$ to $\mathcal{E}'' \to \mathbb{A}^1 \times S$

$Z'' \to \mathbb{A}^1 \times S$
finite

Z'' finite $/ S$
closed in $\mathbb{A}^1_S$

triv. $\cup\, \mathcal{E}'$

compatible by

$\mathbb{A}^1 = \mathbb{A}^1 - Z'' \sqcup W$

$\mathcal{E}'' \xrightarrow{G} \mathbb{A}^1 \times S$

$\mathcal{E}''$ trivial away from finite

By first part $\exists\, V$ of s in S

$\mathcal{E}''|_{\mathbb{A}^1 \times V}$ is trivial

$$\mathcal{E}_u^{\circ}|_{f^{-1}(\mathbb{A}^1 \times V)} = f^* \mathcal{E}'|_{f^{-1}(\mathbb{A}^1 \times V)} = f^*(\mathcal{E}'|_{\mathbb{A}^1 \times V})$$

is trivial

$\mathcal{E}|_{\underset{\substack{\Downarrow \\ x}}{u \cap f^{-1}(\mathbb{A}^1 \times V)}}$ is trivial

Prop. Assumption as in the Thm

$\exists\, u$ and S as in the Thm and

a smooth morphism $u \xrightarrow{p} S$ s.t.

$p|_{u \cap Z}$ is finite [Artin's elementary fibrations]

$$\begin{tikzcd} u \arrow[r] \arrow[dr] & S \times \mathbb{A}^1 \arrow[d] \\ & S \end{tikzcd} \qquad Mor(u, S \times \mathbb{A}^1_{\mathbb{C}}) = Mor(u, S) \times Mor(u, \mathbb{A}^1_{\mathbb{C}})$$

Finite morphism

$\underset{\text{ring}}{A} \to B$ finite if B is finite as an A-module

Affine morphism

$\operatorname{Spec} A \to \operatorname{Spec} B$ finite if $A \to B$ finite

$$\begin{tikzcd} X \arrow[r] & Y \\ \operatorname{Spec} B \arrow[u, hook] \arrow[r, "f'"] & \operatorname{Spec} A \arrow[u, phantom, "\cap"] \end{tikzcd}$$

schemes

(i) $X \times_Y \operatorname{Spec} A$ affine

(ii) f' is finite

(i) $f: X \to Y$ finite $\Rightarrow$ quasi-finite:

$\forall y \in Y$ $f^{-1}(y)$ finite

(ii) f is closed

$\mathcal{U} \underset{\text{open}}{\subset} X$ is quasi-finite

Prop. f quasi-finite & projective $\Rightarrow$ finite

Pf. May assume X is affine, integral (= connected)

$$X \underset{\text{closed}}{\hookrightarrow} \mathbb{A}^N_{\mathbb{C}} \underset{\text{open}}{\hookrightarrow} \mathbb{P}^N_{\mathbb{C}}$$

$\overline{X}$ = closure of X in $\mathbb{P}^N_{\mathbb{C}}$

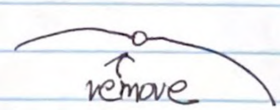
(may not be smooth)

May assume $\overline{X}^{\text{sing}}$ has codim at least 2 in $\overline{X}$

$$\begin{tikzcd} \overline{X} \arrow[r, squiggly] & \overline{X}^{\text{norm}} \arrow[l, bend left, "\text{finite}"] \end{tikzcd}$$ normalization

$(\overline{X}^{\text{norm}})^{\text{sing}}$ has codim ≥ 2

$$\underset{\text{open}}{X} \leftarrow \underset{\text{open}}{X^{\text{norm}}} = X$$



Loops $\quad \mathrm{Maps}(S^1, X)$ in topology

$$\mathrm{Mor}_{\mathbb{C}}(\mathbb{C}^*, X) \qquad \mathbb{C}^* = \mathrm{Spec}\,\mathbb{C}[t^{-1}, t] = \mathrm{Spec}\,\mathbb{C}[t][t^{-1}]$$

$$\mathcal{O} = \mathbb{C}[[t]] = \{a_0 + a_1 t + \cdots + a_n t^n + \cdots\}$$

$$K = \mathbb{C}[[t]](t^{-1}) = \{a_{-N}t^{-N} + a_{1-N}t^{1-N} + \cdots\} = \mathbb{C}((t)) \leftarrow \text{field}$$

$$LX = \mathrm{Mor}_{\mathbb{C}}(\underset{\overset{\|}{\mathring{D}}}{\mathrm{Spec}\,K}, X) \qquad \mathrm{Spec}_{\mathbb{C}}\mathcal{O} = D$$

$\mathring{D}$ punctured formal disc

Ex 1. $X = \mathbb{A}^n_{\mathbb{C}} = \mathrm{Spec}\,\mathbb{C}[x_1, \cdots, x_n]$

$$LX = \mathrm{Mor}_{\mathbb{C}}(\mathbb{C}[x_1, \cdots, x_n], \mathbb{C}((t)))$$

$$= [\mathbb{C}((t))]^n = \{y_1, \cdots, y_n \in \mathbb{C}((t))\} = \{y_i^{(j)}\}$$

Ex 2. $\underset{\overset{\|}{V(t_1, \cdots, t_m)}}{X} \underset{\text{closed}}{\subset} \mathbb{A}^n_{\mathbb{C}} \qquad y_i = \sum_j y_i^{(j)}(t^i)$

$$LX = \{y_1, \cdots, y_n,\ \forall i,\ f_i(y_1, \cdots, y_n) = 0\}$$

$$\underset{\substack{\text{group}\\ \text{closed}}}{G} \subset GL_n = \{(A, d) \in \mathbb{A}^{n^2+1}_{\mathbb{C}},\ \det A \cdot d = 1\}$$

LG is a group

$G = SL_2$

$$LG = \left\{ \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix},\ \begin{matrix} y_{ij} \in \mathbb{C}((t)) \\ y_{11}y_{12} - y_{12}y_{21} = 1 \end{matrix} \right\}$$

$$\begin{pmatrix} t & 1+t^2 \\ 0 & t^{-1} \end{pmatrix} \in LG$$

$$G = GL_1 = G_m = \{ x_1, x_2 : x_1 x_2 = 1 \}$$

$$LG = \{ y_1, y_2 \in \mathbb{C}((t)) : y_1 y_2 = 1 \} = \mathbb{C}((t))^\times$$

$LG : \mathbb{C}\text{-alg} \xrightarrow{\varphi} Set$ ind-scheme $\qquad R[[t]] = \{ \sum r_i t^i \}$

$LG(R) = \underbrace{G(R((t)))}_{\text{group}}$ $\qquad R((t)) = R[[t]][t^{-1}]$

Ex. $LG_m(R) = R((t))^\times$ Ex. describe it
not a field, thus not easy to find invertible element

$L_+ G(R) = G(R[[t]])$ - is a scheme

$$L_+ \mathbb{A}^N_{\mathbb{C}}(R) = R[[t]]^n$$
$$\| $$
$$\mathrm{Spec}\ \mathbb{C}[[y_i^{(j)}, i = 1, \dots N, j \geq 0]]$$

$X \underset{\text{closed}}{\subset} \mathbb{A}^N_{\mathbb{C}}$

SL_2 : $\mathrm{Spec}\, \mathbb{C}[y_{11}^{(j)}, y_{12}^{(j)}, y_{21}^{(j)}, y_{22}^{(j)}]$

$$y_{11}^{(0)} y_{22}^{(0)} - y_{12}^{(0)} y_{21}^{(0)} = 1 \leftarrow (y_{11}y_{22} - y_{12}y_{21} = 1)$$

$$L^+ SL_2 \to SL_2$$

$$\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \mapsto \begin{pmatrix} y_{11}^{(0)} & y_{12}^{(0)} \\ y_{21}^{(0)} & y_{22}^{(0)} \end{pmatrix}$$

Gr_G : $\mathbb{C}\text{-alg}^{op} \to \mathrm{Sets}$

$$R \mapsto \left(G(R((t)))/G(R[[t]])\right)_{\acute{e}t\text{-}sh}$$

$$= LG/L^+G$$

$Gr_G = \varinjlim X^{(i)}$ $X^{(i)}$ are projective schemes

$$X^{(1)} \hookrightarrow X^{(2)} \hookrightarrow \cdots \hookrightarrow X^{(m)} \hookrightarrow \cdots$$

The modular construction

$p \in X$ smooth $\mathbb{C}$-curve

$$Gr_G(R) = \{ (\mathcal{E}, s) \mid \mathcal{E} \xrightarrow{G} X \times \mathrm{Spec}R,\ s : (X - p) \times \mathrm{Spec}R \to \mathcal{E} \}$$

Thm. Two constructions give the same functor

already a sheaf

$$Bun_G(R) = \{ \mathcal{E} \to X \times \mathrm{Spec}R \}$$

without a trivialization is a stack

In our case

$$\begin{tikzcd} \mathcal{E}_\circ \arrow[d, "\iota"'] & \\ \mathcal{E}_\bullet & (X-p)\times \mathrm{Spec} R \arrow[ul, "s"'] \arrow[l, "s"] \end{tikzcd}$$

$$\iota|_{(X-p)\times \mathrm{Spec}R} = id\,(\mathcal{E}|_{(X-p)\times \mathrm{Spec}R})$$

not a stack

Pf. $G(K)/G(\mathcal{O})$ t coordinate near p

$$\gamma \rightsquigarrow \mathcal{E} = \underbrace{G\times(X-p)}_{s} \sqcup G\times D \qquad \gamma \in \mathrm{Aut}(G\times \mathring{D}) = G_K(\mathbb{C})$$

$D \hookrightarrow X \ni p$, $\cup$ open, $X_{aff} = \mathrm{Spec} A$ Dedekind, $p \in X_{aff}$

$$p \rightsquigarrow m \subset A \qquad A/m \simeq \mathbb{C}$$

$$A \hookrightarrow \hat{A} := \varprojlim A/m^n \simeq \mathbb{C}[[t]]$$

$$\begin{tikzcd} X_{aff} & D \arrow[l] \\ X-p & \mathring{D} \arrow[l] \arrow[u, phantom, "\cup"] \end{tikzcd}$$

R. Fedorov
Exotic ....

$$G(\mathcal{O}) = L_+G(\mathbb{C}) = \mathrm{Aut}(X\times D)$$

OTOH $D = \mathrm{Spec}\,\mathbb{C}[[t]]$ criterion of smoothness

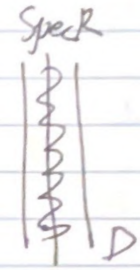
$$\mathbb{C}[[t]] = \varprojlim \mathbb{C}[[t]]/t^n$$

$$\begin{array}{c} \mathcal{E}_D \\ \downarrow \\ D \end{array}$$

~~$LG(R)$~~ $Gr_G \to LG/L_+G$ $\qquad X_R = X \times \mathrm{Spec} R$

$$Gr_G(R) = \{(\mathcal{E}, s) : \mathcal{E} \to X_R,\ s : (X-p)_R \to \mathcal{E}\}$$

$$X_R = (X-p)_R \sqcup_{\mathring{D}_R} D_R$$

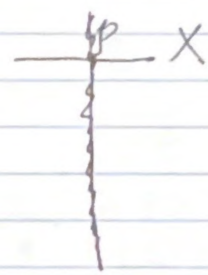


$\mathcal{E}|_{D_R}$ is trivial ?

$\cup$

$\mathrm{Spec} R$

$\mathcal{E}|_{p \times \mathrm{Spec} R}$



Can find $\mathrm{Spec} \tilde{R} \xrightarrow{\text{étale}} \mathrm{Spec} R$

$\mathcal{E}|_{\mathrm{Spec}\tilde{R}}$ is trivial

$\Rightarrow \mathcal{E}|_{\mathrm{Spec}\tilde{R}[[t]]}$ is trivial by continue the process

$$\begin{array}{ccc} (X-p)_{\tilde{R}} & & D_{\tilde{R}} \\ {\scriptstyle s_{\tilde{R}}}\downarrow & & \downarrow \\ \mathcal{E} & & \mathcal{E} \end{array}$$

$$\gamma \in \mathrm{Aut}(\mathring{D}_{\tilde{R}}) = LG(\tilde{R})$$

Now $(\mathcal{E}, s) \in Gr_G(R)$

$$\begin{array}{c} (\mathcal{E}, s) \in Gr_G(R) \\ \downarrow \\ \gamma \in G(\tilde{\hat{R}}) \rightsquigarrow \gamma' \in LG(\tilde{\hat{R}})/L_+G(\tilde{\hat{R}}) \end{array}$$

Claim. Two pullback to

$$LG(\tilde{\hat{R}} \otimes_R \tilde{\hat{R}})/L_+G(\tilde{\hat{R}} \otimes_R \tilde{\hat{R}})$$

are the same

γ' gives a section of $\cancel{Gr_G}$

$(LG/L_+G)_{sh}$

$\blacksquare$

The affine Grassmannian as a presheaf quotient

§ The presheaf quotient suffices

Fix a ring A (e.g. $A=\mathbb{C}$)

a $\underbrace{\text{reductive } A\text{-gp. scheme}}$ G (e.g. $G=GL_n$)

a smooth affine A-gp. scheme whose

geometric A-fiber are connected reductive gps.

Rem. Any reductive A-gp. G becomes split étale

locally on A

Split groups are classified combinatorially

by root data $\mathbb{G}_m^n$, GL_n, SL_n, PGL_n, SO_n, Sp_{2n}, E_6, E_7, E_8, F_4, G_2, $\cdots$

The loop group of G is the function

$$\begin{array}{ccc} LG & \{A\text{-alg's}\} \longrightarrow \{\text{groups}\} \\ \cup & B \longmapsto G(B((t))) \\ & \quad\quad\quad\cup \\ \text{(resp. positive} \to) L^+G & B \longmapsto G(B[[t]]) \end{array}$$

The affine Grassmannian

$$Gr_G : \{A\text{-alg's}\} \to \{\text{pointed sets}\}$$

is the étale sheafification of the presheaf quotient

$$LG/L^+G : \quad B \mapsto G(B((t)))/G(B[[t]])$$

Fact L^+G is an affine A-scheme

(e.g. $L^+G_a : B \mapsto B[[t]]$

is represented by $\mathbb{A}_A^{\mathbb{N}}$)

LG is an ind-affine A-ind-scheme

$$LG = \varinjlim (\underset{\underset{L^+G}{\parallel}}{X_0} \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots)$$

affine A-schemes

Gr_G is an ind-projective A-ind-scheme

at least if $\exists\, G \hookrightarrow GL_{n,A}$

(e.g. if A is a normal domain)

if G not reductive, may not be ind-projective

Main Thm. ($\check{C}$) (a) Gr_G is the Zariski sheafification of the presheaf quotient LG/L^+G

(b) Gr_G is the presheaf quotient if G is totally isotropic in the sense that Zariski locally on A it has a parabolic subgroup that properly needs every factor of G^{ad}.

(split $\Rightarrow$ quasi-split $\Rightarrow$ totally isotropic)

Eg. (of Main Thm. (a))

For a valuation ring $\mathcal{O}$ with function field K and a reductive $\mathcal{O}$-gp. H, we have (from valuative criterion of properness)

$$H(K((t))) = H(V((t)))\,H(K[[t]])$$

e.g. $H = \mathbb{G}_m$ $\quad K((t))^\times = t^{\mathbb{Z}} \cdot K[[t]]^\times$

§ Reinterpretation in terms of torsors

Alternate definition/ modular description interpretation of Gr_G

$Gr_G : B \mapsto \{(E,\tau) \mid E$ is a G-torsor over $B[[t]]$, τ is a trivialization $E|_{B((t))} \simeq G_{B((t))}\}/\sim$

$U \leftarrow (E,\tau)$ with E trivial

LG/L^+G (presheaf quotient)

For the agreement of the two definition of Gr_G, recall $\{G\text{-torsors}/B[[t]]\}/\sim \xrightarrow{\sim} \{G\text{-torsors}/B\}/\sim$ $(*)$

so that $\mathcal{E}$ trivialize over $B'[[t]]$

for some étale faithfully flat $B \Rightarrow B'$

$\rightsquigarrow$ By renaming B to A. Main Thm reduces to

Main Thm' Let G be a reductive group over a ring A. If either

(a) A is semilocal; or

(b) G is totally isotropic;

then no nontrivial G-torsor E over $A[[t]]$ ($\Leftrightarrow$ over A) (*)

trivializes over $A((t))$

$$Gr_G(A) = G(A((t)))/G(A[[t]])$$

Eg. when A is a field, this was known as a consequence of the DVR case of the Grothendieck-Serre conjecture.

In case (b), we have $G(A((t))) = G(A[t^{\pm 1}])\,G(A[[t]])$

so that $Gr_G^{(A)} \cong Gr_G^{alg.}(A) = G(A[t^{\pm 1}])/G(A[t])$

Rem. Not clear how to prove this ~~bijective~~ directly even when $G = GL_n$ and $A = \mathbb{Z}$

§ Passage to $\mathbb{P}^1_A$

By patching with the trivial torsor away from origin over $\mathbb{P}^1_A \setminus \{t=0\}$, reduce Main Thm' to

Thm. (PS. 23, ČF. 24) (a) For a reductive gp. G over a semilocal ring A, every G-torsor E over $\mathbb{P}^1_A$ satisfies $E|_{\{t=\infty\}} \simeq E|_{\{t=0\}}$

($\Leftrightarrow$ s^*E up to iso doesn't depend on $s \in \mathbb{P}^1_A(A)$)

(b) (ČF 24) For a totally isotropic reductive group G over a ring A and a G-torsor E over $\mathbb{P}^1_A$. If $E|_{\{t=\infty\}}$ is trivial, then $E|_{\mathbb{A}^1_A}$ is also trivial.

The proof uses the geometry of $\mathrm{Bun}_G :=$ alg. stack over A that ~~par~~ parametrizes G-torsors over $\mathbb{P}^1_A$

Eg. For an A-gp. M of mult. type have $\mathrm{Bun}_M \cong BM_A \times X_*(M)$ (generalizes, $M=\mathbb{G}_m$ $\mathrm{Pic}(\mathbb{P}^1_A) = \mathrm{Pic}(A) \times \mathbb{Z}$)

(α-inflation of the $\mathbb{G}_m$-torsor $\mathcal{O}(t)$) $\longleftarrow\!\shortmid$ ($\alpha : \overset{\psi}{\mathbb{G}_{m,A}} \twoheadrightarrow M$)

and both (a) and (b) follow for $G = M$

b/c $\mathcal{O}(t)|_{\mathbb{A}^1_A}$ is trivial.

The case of general G uses

Prop. The adjuction map $BG \hookrightarrow \mathrm{Bun}_G$ is an open immersion of alg. stacks $/A$

($\Rightarrow$ If A is local and $E|_{\mathbb{P}^1_A}$ is a G-torsor s.t. $E|_{\mathbb{P}^1_{k(m)}}$ is constant, then E is constant)

In part (b), one reduces to local A by

Lem. (Gabber-Quillen patching) For a locally finitely presented A-gp. scheme H over a ring A, an H-torsor over $\mathbb{A}^1_A$ descends to A iff it does so when base changed to A_m for any maximal ideal $m \subset A$.

$Z(G) \subset G^{sc} \twoheadrightarrow G^{ad}$

Once A is local in (b), reduce to simply connected G using stacky of mult. type gerbes over $\mathbb{P}^1_A$

the proof for sc G uses the Prop. and the following.

Thm. (Borel-Tits) For a simply connected semisimple totally isotropic gp. H over a Henselian DVR $\underbrace{\mathcal{O}}_{k[[t]]}$ with fraction field K, then each $\alpha \in G(K)/G(\mathcal{O})$ is represented by a product of "elementary matrices"

$$L \cdots\cdots LG \qquad B \mapsto \begin{array}{c} G(B((t_1))((t_2))\cdots((t_n))) \\ \cup \\ G(B[[t_1]]\cdots[[t_n]]) \end{array}$$